

# FUNCTORIALITY FOR GENERAL SPIN GROUPS

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**ABSTRACT.** We establish the functorial transfer of generic, automorphic representations from the quasi-split general spin groups to general linear groups over arbitrary number fields, completing an earlier project. Our results are definitive and, in particular, we determine the image of this transfer completely and give a number of applications.

## 1. INTRODUCTION

In this article we complete a project we started in [AS1] by establishing the full transfer of generic, automorphic representations from the quasi-split general spin groups to the general linear group. In particular, we completely determine the image of this transfer.

Our first main result is to establish the transfer of globally generic, automorphic representations from the quasi-split non-split even general spin group,  $\mathrm{GSpin}^*(2n, \mathbb{A}_k)$ , to  $\mathrm{GL}(2n, \mathbb{A}_k)$  (cf. Theorem 3.3). Here,  $k$  denotes an arbitrary number field. We proved the analogous result for the split groups  $\mathrm{GSpin}(2n)$  and  $\mathrm{GSpin}(2n+1)$  in [AS1], but were not able to prove the quasi-split case then since the “stability of root numbers” was not yet available for non-split groups.

Our next main result is to prove that the transferred representation to  $\mathrm{GL}(2n)$ , from either an even or an odd general spin group, is actually an isobaric, automorphic representation (cf. Theorem 5.11).

Our final result gives a complete description of the image of this transfer in terms of  $L$ -functions (cf. Theorem 5.16). This description is exactly what is expected from the theory of twisted endoscopy.

The latter two results allow us to give a number of applications. As a first application, we are able to describe the local component of the transferred representation at the ramified places. In particular, we show that these local components are generic (cf. Proposition 6.1).

Another application is to prove estimates toward the Ramanujan conjecture for the generic spectrum of the general spin groups. We do this by using the best estimates currently known for the general linear groups [LRS]. In particular, our estimates show that if we know the Ramanujan conjecture for  $\mathrm{GL}(m)$  for  $m$  up to  $2n$ , then the Ramanujan conjecture for the generic spectrum of  $\mathrm{GSpin}(2n+1)$  and  $\mathrm{GSpin}(2n)$  follows.

Yet another application of our main results is to give more information about H. Kim’s exterior square transfer from  $\mathrm{GL}(4)$  to  $\mathrm{GL}(6)$  with the help of some recent work of J. Hundley and E. Sayag. We prove that a cuspidal representation  $\Pi$  of  $\mathrm{GL}(6)$  is in the image of Kim’s transfer if and only if the (partial) twisted symmetric square  $L$ -function of  $\Pi$  has a pole at  $s = 1$  (cf. Proposition 6.9).

We now explain our results in more detail. Let  $k$  be a number field and let  $\mathbb{A} = \mathbb{A}_k$  denote its ring of adèles. Let  $\mathbf{G}$  be the split group  $\mathrm{GSpin}(2n+1)$ ,  $\mathrm{GSpin}(2n)$  or one of its quasi-split non-split forms  $\mathrm{GSpin}^*(2n)$  associated with a quadratic extension  $K/k$  (cf. Section 2). There is a natural embedding

$$\iota : {}^L\mathbf{G} \longrightarrow \mathrm{GL}(2n, \mathbb{C}) \times \Gamma_k \tag{1.1}$$

of the  $L$ -group of  $\mathbf{G}$ , as a group over  $k$ , into that of  $\mathrm{GL}(2n)$  described in Section 3. Let  $\pi$  be a globally generic, (unitary) cuspidal, automorphic representation of  $G = \mathbf{G}(\mathbb{A})$ . For almost all places  $v$  of  $k$  the local representation  $\pi_v$  is parametrized by a homomorphism

$$\phi_v : W_v \longrightarrow {}^L\mathbf{G}_v, \quad (1.2)$$

where  $W_v$  is the local Weil group of  $k_v$  and  ${}^L\mathbf{G}_v$  is the  $L$ -group of  $\mathbf{G}$  as a group over  $k_v$ . Langlands Functoriality then predicts that there is an automorphic representation  $\Pi$  of  $\mathrm{GL}(2n, \mathbb{A})$  such that for almost all  $v$ , the local representation  $\Pi_v$  is parametrized by  $\iota \circ \phi_v$ . We established this result for the split case in [AS1]. However, the quasi-split case had to wait because the local technical tools of “stability of  $\gamma$ -factors” (cf. Proposition 3.7) and a result on local  $L$ -functions and normalized intertwining operators (Proposition 3.6) were not available in the quasi-split non-split case. The local result is now available in our cases thanks to the thesis of Wook Kim [WKim] and the stability of  $\gamma$ -factors is available in great generality thanks to a recent work of Cogdell, Piatetski-Shapiro and Shahidi [CPSS1].

As in the split case, the method of proving the existence of an automorphic representation  $\Pi$  is to use converse theorems. This requires knowledge of the analytic properties of the  $L$ -functions for  $\mathrm{GL}(m) \times \mathrm{GL}(2n)$  for  $m \leq 2n - 1$ . The two local tools allow us to relate the  $L$ -functions for  $\mathbf{G} \times \mathrm{GL}$  from the Langlands-Shahidi method to those required in the converse theorems in the following way. Due to the lack of the local Langlands correspondence in general, there is no natural choice for the local components of our candidate representation  $\Pi$  at the finite number of exceptional places of  $k$  where some of our data may be ramified. This means that we have to pick these local representations essentially arbitrarily. However, we show that the local  $L$ - and  $\epsilon$ -factors appearing will become independent of the representation, depending only on the central character, if we twist by a highly ramified character. Globally we can afford to twist our original representation by an idèle class character which is highly ramified at a finite number of places. With this technique we succeed in applying an appropriate version of the converse theorem. The conclusion so far is to have an automorphic representation  $\Pi$  of  $\mathrm{GL}(2n, \mathbb{A})$  which is locally the transfer of  $\pi$  associated with  $\iota$  outside a finite number of places. Moreover, if  $\omega = \omega_\pi$  is the central character of  $\pi$ , then  $\omega_\Pi = \omega^n \mu$ , where  $\mu$  is a quadratic idèle class character, only nontrivial in the quasi-split non-split case.

Next, we get more information about  $\Pi$ . In particular, we prove that  $\Pi$  is indeed an isobaric, automorphic representation (cf. Theorem 5.11 and its corollary). We refer to [L1] for the notion of isobaric representations. For this one needs to know some analytic properties of the Rankin-Selberg type  $L$ -functions  $L(s, \pi \times \tau)$ , where  $\tau$  is a cuspidal representation of  $\mathrm{GL}(m, \mathbb{A})$  and  $\pi$  is a generic representation of  $\mathbf{G}(\mathbb{A})$ . In particular, one needs to know that the  $L$ -function for  $\mathbf{G} \times \mathrm{GL}(m)$  for  $m \leq n$  is holomorphic for  $\Re(s) > 1$  and to know under what conditions this  $L$ -function has a pole at  $s = 1$ . When  $\mathbf{G}$  is a special orthogonal or symplectic group, these results are known thanks to the works of Gelbart, Ginzburg, Piatetski-Shapiro, Rallis and Soudry studying the Rankin-Selberg type zeta integrals giving these  $L$ -functions. For a survey of the methods and results we refer to Soudry’s survey article [Sou2]. In Section 4 we extend this method to the case of  $\mathbf{G} \times \mathrm{GL}$ , where  $\mathbf{G}$  is a general spin group, closely following their method.

Not only does the study of the Rankin-Selberg integral gives us the result on the analytic behavior of the  $L$ -functions, it also provides more information about the image of the transfer. In particular, we prove that  $\Pi \cong \tilde{\Pi} \otimes \omega$ . Here,  $\tilde{\Pi}$  denotes the contragredient of  $\Pi$ . We are also able to prove that the transferred representation  $\Pi$  is unique, it is an isobaric

sum of pairwise inequivalent unitary, cuspidal, automorphic representations

$$\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t, \quad (1.3)$$

and each  $\Pi_i$  satisfies the condition that its twisted symmetric square or twisted exterior square  $L$ -function has a pole at  $s = 1$ , depending on whether we are transferring from even or odd general spin groups (cf. Theorem 5.16).

The automorphic representations  $\Pi$  of  $\mathrm{GL}(2n, \mathbb{A})$  which are transfers from representations  $\pi$  of general spin groups satisfy

$$\Pi \cong \tilde{\Pi} \otimes \omega, \quad (1.4)$$

as predicted by the theory of twisted endoscopy [KoSh]. In fact, these representations comprise precisely the image of the transfer. While we prove half of this statement we note that the other half of this, i.e., the fact that any representation of  $\mathrm{GL}(2n, \mathbb{A})$  satisfying (1.4) is a transfer from a representation of a general spin group has now also been proved thanks to the work of J. Hundley and E. Sayag, extending the descent theory results of Ginzburg, Rallis, and Soudry from the case of classical groups ( $\omega = 1$ ) to our case.

If a representation  $\Pi$  of  $\mathrm{GL}(2n, \mathbb{A})$  satisfies (1.4), then

$$L^T(s, \Pi \times (\Pi \otimes \omega^{-1})) = L^T(s, \Pi, \mathrm{Sym}^2 \otimes \omega^{-1}) L^T(s, \Pi, \wedge^2 \otimes \omega^{-1}), \quad (1.5)$$

where  $T$  is a sufficiently large finite set of places of  $k$  and  $L^T$  denotes the product over  $v \notin T$  of the local  $L$ -functions. The  $L$ -function on the left hand side of (1.5) has a pole at  $s = 1$ , which implies that one, and only one, of the two  $L$ -functions on the right hand side of (1.5) has a pole at  $s = 1$ . If the twisted exterior square  $L$ -function has a pole at  $s = 1$ , then  $\Pi$  is a transfer from an odd general spin group and if the twisted symmetric square  $L$ -function has a pole at  $s = 1$ , then  $\Pi$  is a transfer from an even general spin group (which may be split or quasi-split non-split).

To tell the split and quasi-split non-split cases apart note that from (1.4) we have

$$\omega_\Pi^2 = \omega^{2n}. \quad (1.6)$$

In other words,  $\mu = \omega_\Pi \omega^{-n}$  is a quadratic idèle class character. If  $\mu$  is the trivial character, then  $\Pi$  is the transfer of a generic representation of even split general spin group and if  $\mu$  is a nontrivial quadratic character, then  $\Pi$  is a transfer from a generic representation of a quasi-split group associated with the quadratic extension of  $k$  determined by  $\mu$  through class field theory.

Our results here along with those of Hundley and Sayag [HS1, HS2, HS3] give a complete description of the image of the transfer for the generic representations of the general spin groups. It remains to study the transfers of non-generic, cuspidal, automorphic representations of the general spin groups, which our current method cannot handle. However, the image of the generic spectrum is conjecturally the full image of the tempered spectrum, generic or not, of the general spin groups since each tempered  $L$ -packet is expected to include a generic member. We refer to [Sh6] for more details on this conjecture. We point out that Arthur's upcoming book [Ar] would answer this question in the case of classical groups. However, his book does not cover the case of general spin groups.

We can apply our results in this paper, along with those of Cogdell, Kim, Krishnamurthy, Piatetski-Shapiro, and Shahidi for the classical and unitary groups, to give some uniform results on reducibility of local induced representations of non-exceptional  $p$ -adic groups. We will address this question along with other local applications of generic functoriality in a forthcoming paper [ACS].

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## 2. THE PRELIMINARIES

Let  $k$  be a number field and let  $\mathbb{A} = \mathbb{A}_k$  be the ring of adèles of  $k$ . Let  $n \geq 0$  be an integer. We consider the general spin groups. The group  $\mathrm{GSpin}(2n+1)$  is a split connected reductive group of type  $B_n$  defined over  $k$  whose dual group is  $\mathrm{GSp}(2n, \mathbb{C})$ . Similarly, the split connected reductive group  $\mathrm{GSpin}(2n)$  over  $k$  is of type  $D_n$  and its dual is isomorphic to  $\mathrm{GSO}(2n, \mathbb{C})$ , the connected component of the group  $\mathrm{GO}(2n, \mathbb{C})$ . There are also quasi-split non-split groups  $\mathrm{GSpin}^*(2n)$  in the even case. They are of type  ${}^2D_n$  and correspond to quadratic extensions of  $k$ . A more precise description is given below. We also refer to [CPSS2, §7 & 1] for a review of the generalities about these groups.

We fix a Borel subgroup  $\mathbf{B}$  and a Cartan subgroup  $\mathbf{T} \subset \mathbf{B}$ . The associated based root datum to  $(\mathbf{B}, \mathbf{T})$  will be denoted by  $(X, \Delta, X^\vee, \Delta^\vee)$  which we further explicate below. Our choice of the notation for the root data below is consistent with the Bourbaki notation [Bou].

**2.1. Structure of GSpin Groups.** We describe the odd and even GSpin groups by introducing a based root datum for each as in [Spr, §7.4.1]. A more detailed description can also be found in [AS1, §2]. We use these data as our tool to work with the groups in question due to the lack of a convenient matrix representation.

**2.1.1. The root datum of GSpin(2n + 1).** The root datum of GSpin(2n + 1) is given by  $(X, R, X^\vee, R^\vee)$ , where  $X$  and  $X^\vee$  are  $\mathbb{Z}$ -modules generated by generators  $e_0, e_1, \dots, e_n$  and  $e_0^*, e_1^*, \dots, e_n^*$ , respectively. The pairing

$$\langle \cdot, \cdot \rangle : X \times X^\vee \longrightarrow \mathbb{Z} \quad (2.1)$$

is the standard pairing. Moreover, the roots and coroots are given by

$$R = R_{2n+1} = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\} \quad (2.2)$$

$$R^\vee = R_{2n+1}^\vee = \{\pm(e_i^* - e_j^*) : 1 \leq i < j \leq n\} \cup \{\pm(e_i^* + e_j^* - e_0^*) : 1 \leq i < j \leq n\} \cup \{\pm(2e_i^* - e_0^*) \mid 1 \leq i \leq n\} \quad (2.3)$$

along with the bijection  $R \longrightarrow R^\vee$  given by

$$(\pm(e_i - e_j))^\vee = \pm(e_i^* - e_j^*) \quad (2.4)$$

$$(\pm(e_i + e_j))^\vee = \pm(e_i^* + e_j^* - e_0^*) \quad (2.5)$$

$$(\pm e_i)^\vee = \pm(2e_i^* - e_0^*). \quad (2.6)$$

It is easy to verify that the conditions (RD 1) and (RD 2) of [Spr, §7.4.1] hold. Moreover, we fix the following choice of simple roots and coroots:

$$\Delta = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_n\}, \quad (2.7)$$

$$\Delta^\vee = \{e_1^* - e_2^*, e_2^* - e_3^*, \dots, e_{n-1}^* - e_n^*, 2e_n^* - e_0^*\}. \quad (2.8)$$

This datum determines the group GSpin(2n + 1) uniquely, equipped with a Borel subgroup containing a maximal torus.

**2.1.2. The root datum of GSpin(2n).** Next, we give a similar description for the even case. The root datum of GSpin(2n) is given by  $(X, R, X^\vee, R^\vee)$  where  $X$  and  $X^\vee$  and the pairing is as above and the roots and coroots are given by

$$R = R_{2n} = \{\pm(e_i \pm e_j) : 1 \leq i < j \leq n\} \quad (2.9)$$

$$R^\vee = R_{2n}^\vee = \{\pm(e_i^* - e_j^*) : 1 \leq i < j \leq n\} \cup \{\pm(e_i^* + e_j^* - e_0^*) : 1 \leq i < j \leq n\} \quad (2.10)$$

along with the bijection  $R \longrightarrow R^\vee$  given by

$$(\pm(e_i - e_j))^\vee = \pm(e_i^* - e_j^*) \quad (2.11)$$

$$(\pm(e_i + e_j))^\vee = \pm(e_i^* + e_j^* - e_0^*). \quad (2.12)$$

It is easy again to verify that the conditions (RD 1) and (RD 2) of [Spr, §7.4.1] hold. Similar to the odd case we fix the following choice of simple roots and coroots:

$$\Delta = \{e_1 - e_2, e_2 - e_3, \dots, e_{n-1} - e_n, e_{n-1} + e_n\}, \quad (2.13)$$

$$\Delta^\vee = \{e_1^* - e_2^*, e_2^* - e_3^*, \dots, e_{n-1}^* - e_n^*, e_{n-1}^* + e_n^* - e_0^*\}. \quad (2.14)$$

This based root datum determines the group GSpin(2n) uniquely, equipped with a Borel subgroup containing a maximal torus.

**2.1.3. The quasi-split forms of  $\mathrm{GSpin}(2n)$ .** In the even case, quasi-split non-split forms also exist. We fix a splitting  $(\mathbf{B}, \mathbf{T}, \{x_\alpha\}_{\alpha \in \Delta})$ , where  $\{x_\alpha\}$  is a collection of root vectors, one for each simple root of  $\mathbf{T}$  in  $\mathbf{B}$ . As explained in [CPSS2, §7.1] for the quasi-split forms of  $\mathrm{SO}(2n)$ , the quasi-split forms of  $\mathrm{GSpin}(2n)$  over  $k$  are in one-one correspondence with homomorphisms from  $\mathrm{Gal}(\bar{k}/k)$  to the group of automorphisms of the character lattice preserving  $\Delta$ . This group has two elements: the trivial and the one switching  $e_{n-1} - e_n$  and  $e_{n-1} + e_n$  while keeping all other simple roots fixed.

By Class Field Theory such homomorphisms correspond to quadratic characters

$$\mu : k^\times \backslash \mathbb{A}_k^\times \longrightarrow \{\pm 1\}.$$

When  $\mu$  is nontrivial we denote the associated quasi-split non-split group with  $\mathrm{GSpin}^\mu(2n)$  or simply  $\mathrm{GSpin}^*(2n)$  when the particular  $\mu$  is unimportant. We will also denote the quadratic extension of  $k$  associated with  $\mu$  by  $K^\mu/K$  or simply  $K/k$ .

**2.2. Embeddings.** In order to proceed with the analogues of the relevant integrals for  $\mathrm{GSpin}$  groups we need certain embeddings of these groups inside each other, which we now review. We first recall some basic facts about linear algebraic groups.

Let  $\mathbf{G}$  be a connected reductive linear algebraic group over  $k$  with a fixed Borel subgroup containing a fixed maximal torus. Denote the associated roots by  $R$  and the positive roots by  $R^+$ . For each  $\alpha \in R$  denote the root group homomorphism associated with  $\alpha$  by

$$u_\alpha : \mathbb{G}_a \longrightarrow \mathbf{G}$$

and denote the root group by  $\mathbf{U}_\alpha$ , the image of  $u_\alpha$  in  $\mathbf{G}$ . We are now prepared to describe the various embeddings we will need.

**2.2.1. Embedding  $\mathbf{i} : \mathrm{GSpin}(2m+1) \hookrightarrow \mathrm{GSpin}(2n)$  for  $m < n$ .** The group  $\mathrm{GSpin}(2m+1)$  is an *almost direct* product (i.e., with finite intersection) of  $\mathrm{Spin}(2m+1)$ , the derived group of  $\mathrm{GSpin}(2m+1)$ , and the connected component of its center, which is a torus. The derived group is generated by the root subgroups and, by [AS1, Prop. 2.3], the connected component of the center equals

$$\{e_0^*(t) \mid t \in \mathrm{GL}(1)\}.$$

To describe the embedding we embed each root subgroup of  $\mathrm{GSpin}(2m+1)$  into  $\mathrm{GSpin}(2n)$  and also embed the connected component of the center of the former into the latter. We should, however, ensure that the images of elements in the intersection are consistent.

Let us use the notation  $e_i$  and  $e_i^*$  for roots and coroots of  $\mathrm{GSpin}(2m+1)$  and  $f_i$  and  $f_i^*$  for those of  $\mathrm{GSpin}(2n)$ . By (2.2) the roots of  $\mathrm{GSpin}(2m+1)$  are given by

$$R_{2m+1} = \{\pm e_i \mid 1 \leq i \leq m\} \cup \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq m\}$$

and by (2.9) those of  $\mathrm{GSpin}(2n)$  may be written as

$$R_{2n} = \{\pm(f_i \pm f_j) \mid 1 \leq i < j \leq n\}.$$

For  $1 \leq i < j \leq m$ , we define

$$\mathbf{i}(u_{e_i - e_j}(x)) = u_{f_i - f_j}(x) \tag{2.15}$$

$$\mathbf{i}(u_{e_i + e_j}(x)) = u_{f_i + f_j}(x) \tag{2.16}$$

and for  $1 \leq i \leq m$ , we define

$$\mathbf{i}(u_{e_i}(x)) = u_{f_i - f_n}(x) u_{f_i + f_n}(-x). \tag{2.17}$$

For negative roots we define  $\mathbf{i}$  in a similar way using the corresponding negative roots on the right hand side. Also, set

$$\mathbf{i}(e_0^*(t)) = f_0^*(t). \tag{2.18}$$

**Lemma 2.19.** *For  $m < n$  the embedding  $i : \mathrm{GSpin}(2m+1) \hookrightarrow \mathrm{GSpin}(2n)$  defined via (2.15)–(2.18) is well-defined.*

*Proof.* We have to check that  $i$  is well-defined on the intersection of the derived group and the connected component of the center.

As verified in [AS1, §2] the intersection consists of the trivial element and the nontrivial element  $c = \alpha_m^\vee(-1)$ . On the one hand we have

$$\alpha_m^\vee(-1) = (2e_m^* - e_0^*)(-1) = e_0^*(-1)^{-1} = e_0^*(-1)$$

and consequently

$$i(c) = f_0^*(-1).$$

On the other hand  $c$  also belongs to the derived group and can be expressed in terms of the root group homomorphisms as follows. Recall [AS1, Eq. (18)–(20)] that for any root  $\alpha$  we have

$$\alpha^\vee(-1) = w_\alpha^2, \quad (2.20)$$

where  $w_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ . This means that

$$\begin{aligned} i(w_{\alpha_m}) &= i(u_{e_m}(1)u_{-e_m}(-1)u_{e_m}(1)) \\ &= u_{f_m-f_n}(1)u_{f_m+f_n}(-1) \cdot u_{-f_m+f_n}(-1)u_{-f_m-f_n}(1) \cdot u_{f_m-f_n}(1)u_{f_m+f_n}(-1) \\ &= u_{f_m-f_n}(1)u_{-f_m+f_n}(-1)u_{f_m-f_n}(1) \cdot u_{f_m+f_n}(-1)u_{-f_m-f_n}(1)u_{f_m+f_n}(-1) \\ &= w_{f_m-f_n}w_{f_m+f_n}^{-1}. \end{aligned}$$

Hence,

$$\begin{aligned} i(c) &= i(w_{\alpha_m})^2 \\ &= (w_{f_m-f_n}w_{f_m+f_n}^{-1})^2 \\ &= (f_m - f_n)^\vee(-1) \cdot (-f_m - f_n)^\vee(-1) \\ &= (f_m^* - f_n^* - f_m^* - f_n^* + f_0^*)(-1) \\ &= f_0^*(-1). \end{aligned}$$

Here, we have used (2.11), (2.12), and the fact that  $w_{f_m-f_n}$  and  $w_{f_m+f_n}$  commute [AS1, p. 157]. We conclude that  $i(c)$  is well-defined and this proves the lemma.  $\square$

**2.2.2. Embedding  $i : \mathrm{GSpin}(2m) \hookrightarrow \mathrm{GSpin}(2n+1)$  for  $m \leq n$ .** We proceed in a similar way for this embedding as well. Recall that again the group  $\mathrm{GSpin}(2m)$  is an almost direct product of its derived group and the connected component of the center

$$\{e_0^*(t) \mid t \in \mathrm{GL}(1)\}.$$

Using a similar notation as before recall that as in (2.2) and (2.9) the roots of  $\mathrm{GSpin}(2m)$  and  $\mathrm{GSpin}(2n+1)$  are respectively given by

$$R_{2m} = \{\pm(e_i \pm e_j) \mid 1 \leq i < j \leq m\}.$$

and

$$R_{2n+1} = \{\pm f_i \mid 1 \leq i \leq n\} \cup \{\pm(f_i \pm f_j) \mid 1 \leq i < j \leq n\}.$$

For  $1 \leq i < j \leq m$  we define

$$i(u_{e_i-e_j}(x)) = u_{f_i-f_j}(x) \quad (2.21)$$

$$i(u_{e_i+e_j}(x)) = u_{f_i+f_j}(x). \quad (2.22)$$

For negative roots we define  $i$  in a similar way using the corresponding negative roots on the right hand side. Also, set

$$i(e_0^*(t)) = f_0^*(t). \quad (2.23)$$

**Lemma 2.24.** *For  $m \leq n$  the embedding  $i : \mathrm{GSpin}(2m) \hookrightarrow \mathrm{GSpin}(2n+1)$  defined via (2.21)–(2.23) is well-defined.*

*Proof.* We have to check that  $i$  is well-defined on the intersection of the derived group and the connected component of the center. This intersection again consists of two elements with the nontrivial element now being  $c = \alpha_{m-1}^\vee(-1)\alpha_m^\vee(-1)$  [AS1, §2].

On the one hand we have

$$\alpha_{m-1}^\vee(-1)\alpha_m^\vee(-1) = (2e_{m-1}^* - e_0^*)(-1) = e_0^*(-1)^{-1} = e_0^*(-1)$$

and consequently

$$i(c) = f_0^*(-1).$$

On the other hand  $c$  also belongs to the derived group. We have

$$\begin{aligned} i(w_{\alpha_{m-1}} w_{\alpha_m}) &= i(u_{e_{m-1}-e_m}(1)u_{-e_{m-1}+e_m}(-1)u_{e_{m-1}-e_m}(1) \\ &\quad \cdot u_{e_{m-1}+e_m}(1)u_{-e_{m-1}-e_m}(-1)u_{e_{m-1}-e_m}(1)) \\ &= u_{f_{m-1}-f_m}(1)u_{-f_{m-1}+f_m}(-1)u_{f_{m-1}-f_m}(1) \\ &\quad \cdot u_{f_{m-1}+f_m}(1)u_{-f_{m-1}-f_m}(-1)u_{f_{m-1}-f_m}(1) \\ &= w_{f_{m-1}-f_m} w_{f_{m-1}+f_m}. \end{aligned}$$

Hence,

$$\begin{aligned} i(c) &= i(w_{\alpha_{m-1}}^2 w_{\alpha_m}^2) \\ &= (f_{m-1} - f_m)^\vee(-1) \cdot (f_{m-1} + f_m)^\vee(-1) \\ &= (f_{m-1}^* - f_m^* + f_{m-1}^* + f_m^* - f_0^*)(-1) \\ &= f_0^*(-1). \end{aligned}$$

Here, we have used (2.4)–(2.6), and the fact that  $w_{f_m-f_n}$  and  $w_{f_m+f_n}$  commute [AS1, p. 157]. We conclude that  $i(c)$  is well-defined and this proves the lemma.  $\square$

We end this section by proving a lemma which explicitly gives the image under the map  $i$  of elements in the maximal torus of  $\mathbf{H}$ . We will use this lemma later when we discuss convergence of certain zeta integrals we have to deal with.

**Lemma 2.25.** *With notation as above, let  $a = e_0^*(t_0)e_1^*(t_1)\cdots e_m^*(t_m)$  be an arbitrary element in the maximal torus of  $\mathbf{H}$ . Then,*

$$i(a) = \begin{cases} f_0^*(t_0)f_1^*(t_1)\cdots f_{m-1}^*(t_{m-1})f_m^*(t_m) & \text{if } \mathbf{H} \text{ is even and } \mathbf{G} \text{ is odd,} \\ f_0^*(-t_0)f_1^*(t_1)\cdots f_{m-1}^*(t_{m-1})f_m^*(-t_m)f_n^*(-1) & \text{if } \mathbf{H} \text{ is odd and } \mathbf{G} \text{ is even.} \end{cases}$$

*Proof.* The proof is essentially a careful chasing of the definitions in terms of the root data. First, assume that  $\mathbf{H} = \mathrm{GSpin}(2m)$ . We have

$$\begin{aligned} a &= e_0^* \left( t_0(t_1 \cdots t_{m-1}t_m)^{1/2} \right) (e_1^* - e_2^*)(t_1)(e_2^* - e_3^*)(t_1t_2) \cdots (e_{m-2}^* - e_{m-1}^*)(t_1t_2 \cdots t_{m-2}) \\ &\quad (e_{m-1}^* - e_m^*) \left( (t_1 \cdots t_{m-1}t_m^{-1})^{1/2} \right) (e_{m-1}^* + e_m^* - e_0^*) \left( (t_1 \cdots t_{m-1}t_m)^{1/2} \right), \end{aligned}$$

where the choices of the square roots have to be made appropriately. More precisely, in order to get  $e_{m-1}^*(t_{m-1})$  and  $e_m^*(t_m)$  we have to choose the square roots in the last two



terms consistently. Writing  $(t_1 \cdots t_{m-1} t_m^{-1})^{1/2} = (t_1 \cdots t_{m-1})^{1/2} t_m^{-1/2}$ , we make arbitrary choices for the square roots  $(t_1 \cdots t_{m-1})^{1/2}$  and  $t_m^{-1/2}$  in the term  $e_{m-1}^* - e_m^*$  and use the same choices of the square roots for  $(t_1 \cdots t_{m-1})^{1/2} t_m^{1/2} = (t_1 \cdots t_{m-1} t_m)^{1/2}$  in the last term  $e_{m-1}^* + e_m^* - e_0^*$ . Now, in order to get  $e_0^*(t_0)$  the choice of the square root in the first term  $e_0^*$  has to be the same as that in the last term.

Applying the map  $\mathbf{i}$  and using the root data details we get

$$\begin{aligned} \mathbf{i}(a) &= f_0^* \left( t_0 (t_1 \cdots t_{m-1} t_m)^{1/2} \right) (f_1 - f_2)^\vee(t_1) \cdots (f_{m-2} - f_{m-1})^\vee(t_1 \cdots t_{m-1}) \\ &\quad (f_{m-1} - f_m)^\vee \left( (t_1 \cdots t_{m-1} t_m^{-1})^{1/2} \right) (f_{m-1} + f_m)^\vee \left( (t_1 \cdots t_{m-1} t_m)^{1/2} \right) \\ &= f_0^* \left( t_0 (t_1 \cdots t_{m-1} t_m)^{1/2} \right) (f_1^* - f_2^*)(t_1) \cdots (f_{m-2}^* - f_{m-1}^*)(t_1 t_2 \cdots t_{m-2}) \\ &\quad (f_{m-1}^* - f_m^*) \left( (t_1 \cdots t_{m-1} t_m^{-1})^{1/2} \right) (f_{m-1}^* + f_m^* - f_0^*) \left( (t_1 \cdots t_{m-1} t_m)^{1/2} \right) \\ &= f_0^*(t_0) f_1^*(t_1) \cdots f_{m-1}^*(t_{m-1}) f_m^*(t_m) \end{aligned}$$

Again, similar consistent choices of the square roots need to be made.

Next, assume that  $\mathbf{H} = \mathrm{GSpin}(2m+1)$ . A similar calculation will go through. A necessary step, however, is to calculate the image under  $\mathbf{i}$  of  $e_m^\vee(x)$ . To do this, note that

$$e_m^\vee(x) = w_{e_m}(x) w_{e_m}(1)^{-1} = u_{e_m}(x) u_{-e_m}(-x^{-1}) u_{e_m}(x) (u_{e_m}(1) u_{-e_m}(-1) u_{e_m}(1))^{-1}.$$

Apply  $\mathbf{i}$  to get

$$\begin{aligned} \mathbf{i}(e_m^\vee(x)) &= u_{f_m-f_n}(x) u_{f_m+f_n}(-x) \cdot u_{-f_m+f_n}(-x^{-1}) u_{-f_m-f_n}(x^{-1}) \cdot u_{f_m-f_n}(x) u_{f_m+f_n}(-x) \\ &\quad (u_{f_m-f_n}(1) u_{f_m+f_n}(-1) \cdot u_{-f_m+f_n}(-1) u_{-f_m-f_n}(1) \cdot u_{f_m-f_n}(1) u_{f_m+f_n}(-1))^{-1} \\ &= u_{f_m-f_n}(x) u_{-f_m+f_n}(-x^{-1}) u_{f_m-f_n}(x) (u_{f_m-f_n}(1) u_{-f_m+f_n}(-1) u_{f_m-f_n}(1))^{-1} \\ &\quad u_{f_m+f_n}(-x) u_{-f_m-f_n}(x^{-1}) u_{f_m+f_n}(-x) (u_{f_m+f_n}(-1) u_{-f_m-f_n}(1) u_{f_m+f_n}(-1))^{-1} \\ &= w_{f_m-f_n}(x) w_{f_m-f_n}(1)^{-1} \cdot w_{f_m+f_n}(-x) w_{f_m+f_n}(-1)^{-1} \\ &= (f_m - f_n)^\vee(x) \cdot (f_m + f_n)^\vee(-x). \end{aligned}$$

This last calculation is responsible for the appearance of the negative signs in the  $\mathbf{H}$  odd case in the statement of the lemma.  $\square$

### 3. WEAK TRANSFER FOR THE QUASI-SPLIT $\mathrm{GSpin}(2n)$

In this section  $\mathbf{G} = \mathrm{GSpin}^*(2n)$  will denote one of the quasi-split non-split forms of  $\mathrm{GSpin}(2n)$  as in 2.1.3. We will denote the associated quadratic extension by  $K/k$  and  $\mathbb{A} = \mathbb{A}_k$  will continue to denote the ring of adèles of  $k$ . Also,  $\mathbf{G}$  is associated with a nontrivial quadratic character  $\mu : k^\times \backslash \mathbb{A}_k^\times \rightarrow \{\pm 1\}$ .

The connected component of the  $L$ -group of  $\mathbf{G}$  is  ${}^L\mathbf{G}^0 = \mathrm{GSO}(2n, \mathbb{C})$  and the  $L$ -group can be written as

$${}^L\mathbf{G} = \mathrm{GSO}(2n, \mathbb{C}) \rtimes W_k,$$

where the Weil group acts through the quotient

$$W_k/W_K \cong \mathrm{Gal}(K/k).$$

The  $L$ -group of  $\mathrm{GL}(2n)$  is  $\mathrm{GL}(2n, \mathbb{C}) \times W_k$ , a direct product because  $\mathrm{GL}(2n)$  is split. These are the Weil forms of the  $L$ -group, or we can equivalently use the Galois forms of the  $L$ -groups.

We define a map

$$\iota : \mathrm{GSO}(2n, \mathbb{C}) \rtimes \Gamma_k \longrightarrow \mathrm{GL}(2n, \mathbb{C}) \times \Gamma_k \quad (3.1)$$

$$(g, \gamma) \mapsto \begin{cases} (g, \gamma) & \text{if } \gamma|_K = 1, \\ (hgh^{-1}, \gamma) & \text{if } \gamma|_K \neq 1, \end{cases} \quad (3.2)$$

where  $\gamma \in \Gamma_k$ ,  $g \in \mathrm{GSO}(2n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{C})$ , and  $h = h^{-1}$  is the matrix

$$\begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}.$$

(We refer to [CPSS2, §7.1] for more details.) The map  $\iota$  is an  $L$ -homomorphism. We also have a compatible family of local  $L$ -homomorphisms  $\iota_v : {}^L\mathbf{G}_v \longrightarrow \mathrm{GL}(2n, \mathbb{C}) \times W_v$ . Our purpose in this section is to prove the existence of a weak transfer of globally generic, cuspidal, automorphic representations of  $G = \mathbf{G}(\mathbb{A})$  to automorphic representations of  $\mathrm{GL}(2n, \mathbb{A})$  associated with  $\iota$ .

**Theorem 3.3.** *Let  $K/k$  be a quadratic extension of number fields and let  $\mathbf{G} = \mathrm{GSpin}^*(2n)$  be as above. Let  $\psi$  be a nontrivial continuous additive character of  $k \backslash \mathbb{A}_k$ . The choice of  $\psi$  and the splitting above defines a non-degenerate additive character of  $\mathbf{U}(k) \backslash \mathbf{U}(\mathbb{A})$ , again denoted by  $\psi$ .*

*Let  $\pi = \otimes_v \pi_v$  be an irreducible, globally  $\psi$ -generic, cuspidal, automorphic representation of  $G = \mathbf{G}(\mathbb{A}_k)$ . Write  $\psi = \otimes_v \psi_v$ . Let  $S$  be a nonempty finite set of non-archimedean places  $v$  of  $k$  such that for every non-archimedean  $v \notin S$  both  $\pi_v$  and  $\psi_v$ , as well as  $K_w/k_v$  for  $w|v$ , are unramified. Then there exists an automorphic representation  $\Pi = \otimes_v \Pi_v$  of  $\mathrm{GL}(2n, \mathbb{A}_k)$  such that for all  $v \notin S$  the homomorphism parametrizing the local representation  $\Pi_v$  is given by*

$$\Phi_v = \iota_v \circ \phi_v : W_{k_v} \rightarrow \mathrm{GL}(2n, \mathbb{C}),$$

*where  $W_{k_v}$  denotes the local Weil group of  $k_v$  and  $\phi_v : W_{k_v} \longrightarrow {}^L\mathbf{G}$  is the homomorphism parametrizing  $\pi_v$ . Moreover, if  $\omega_\Pi$  and  $\omega_\pi$  denote the central characters of  $\Pi$  and  $\pi$ , respectively, then  $\omega_\Pi = \omega_\pi^n \mu$ , where  $\mu$  is a nontrivial quadratic idèle class character. Furthermore,  $\Pi$  and  $\widehat{\Pi} \otimes \omega_\pi$  are nearly equivalent.*

*Remark 3.4.* We proved an analogous result for the split groups  $\mathrm{GSpin}(2n+1)$  and  $\mathrm{GSpin}(2n)$  in [AS1].

To prove the theorem we will use a suitable version of the converse theorems of Cogdell and Piatetski-Shapiro [CPS1, CPS2]. The exact version we need can be found in [CPSS2, §2] which we quickly review below. Next we introduce an irreducible, admissible representation  $\Pi$  of  $\mathrm{GL}(2n, \mathbb{A})$  as a candidate for the transfer of  $\pi$ . We then prove that  $\Pi$  satisfies the required conditions of the converse theorem and hence is automorphic. Along the way we also verify the remaining properties of  $\Pi$  stated in Theorem 3.3.

**3.1. The Converse Theorem.** Let  $k$  be a number field and fix a non-empty finite set  $S$  of non-archimedean places of  $k$ . For each integer  $m$  let

$$\mathcal{A}_0(m) = \{\tau \mid \tau \text{ is a cuspidal representation of } \mathrm{GL}(m, \mathbb{A}_k)\}$$

and

$$\mathcal{A}_0^S(m) = \{\tau \in \mathcal{A}_0(m) \mid \tau_v \text{ is unramified for all } v \in S\}.$$

Also, for a positive integer  $N$  let

$$\mathcal{T}(N-1) = \prod_{m=1}^{N-1} \mathcal{A}_0(m) \quad \text{and} \quad \mathcal{T}^S(N-1) = \prod_{m=1}^{N-1} \mathcal{A}_0^S(m)$$

and for  $\eta$  a continuous character of  $k^\times \backslash \mathbb{A}_k^\times$  let

$$\mathcal{T}(S; \eta) = \mathcal{T}^S(N-1) \otimes \eta = \{ \tau = \tau' \otimes \eta \mid \tau' \in \mathcal{T}^S(N-1) \}.$$

For our purposes we will apply the following theorem with  $N = 2n$ .

**Theorem 3.5.** (*Converse theorem of Cogdell and Piatetski-Shapiro*) *Let  $\Pi = \otimes_v \Pi_v$  be an irreducible, admissible representation of  $\mathrm{GL}(N, \mathbb{A}_k)$  whose central character  $\omega_\Pi$  is invariant under  $k^\times$  and whose  $L$ -function*

$$L(s, \Pi) = \prod_v L(s, \Pi_v)$$

*is absolutely convergent in some right half plane. Let  $S$  be a finite set of non-archimedean places of  $k$  and let  $\eta$  be a continuous character of  $k^\times \backslash \mathbb{A}_k^\times$ . Suppose that for every  $\tau \in \mathcal{T}(S; \eta)$  the  $L$ -function  $L(s, \tau \times \Pi)$  is nice, i.e., it satisfies the following three conditions:*

- (1)  $L(s, \tau \times \Pi)$  and  $L(s, \tilde{\tau} \times \tilde{\Pi})$  extend to entire functions of  $s \in \mathbb{C}$ .
- (2)  $L(s, \tau \times \Pi)$  and  $L(s, \tilde{\tau} \times \tilde{\Pi})$  are bounded in vertical strips.
- (3) The functional equation  $L(s, \tau \times \Pi) = \epsilon(s, \tau \times \Pi) L(s, \tilde{\tau} \times \tilde{\Pi})$  holds.

*Then there exists an automorphic representation  $\Pi'$  of  $\mathrm{GL}(N, \mathbb{A}_k)$  such that  $\Pi_v \cong \Pi'_v$  for all  $v \notin S$ .  $\square$*

The twisted  $L$ - and  $\epsilon$ -factors in the statement are those in [CPS1]. In particular, they are Artin factors and known to be the same as the ones coming from the Langlands-Shahidi method at all places.

**3.2.  $L$ -functions for  $\mathrm{GL}(m) \times \mathrm{GSpin}^*(2n)$ .** Let  $\pi$  be an irreducible, admissible, globally generic representation of  $\mathrm{GSpin}^*(2n, \mathbb{A}_k)$  and let  $\tau$  be a cuspidal representation of  $\mathrm{GL}(m, \mathbb{A}_k)$  with  $m \geq 1$ . The group  $\mathrm{GSpin}^*(2(m+n))$  has a standard maximal Levi  $\mathrm{GL}(m) \times \mathrm{GSpin}^*(2n)$  and we have the completed  $L$ -functions

$$L(s, \tau \times \pi) = \prod_v L(s, \tau_v \times \pi_v) = \prod_v L(s, \tau_v \otimes \pi_v, \iota'_v \otimes \iota_v) = L(s, \tau \otimes \pi, \iota' \otimes \iota),$$

with similar  $\epsilon$ - and  $\gamma$ -factors, defined via the Langlands-Shahidi method in [Sh3]. Here,  $\iota$  is the representation of the  $L$ -group of  $\mathrm{GSpin}^*(2n)$  we described before and  $\iota'$  is the projection map onto the first factor in the  $L$ -group  ${}^L\mathrm{GL}(m) = \mathrm{GL}(m, \mathbb{C}) \times W_k$ .

**Proposition 3.6.** *Let  $S$  be a non-empty finite set of finite places of  $k$  and let  $\eta$  be a character of  $k^\times \backslash \mathbb{A}_k^\times$  such that, for some  $v \in S$ ,  $\eta_v^2$  is ramified. Then for all  $\tau \in \mathcal{T}(S; \eta)$  the  $L$ -function  $L(s, \tau \times \pi)$  is nice, i.e., it satisfies the following three conditions:*

- (1)  $L(s, \tau \times \pi)$  and  $L(s, \tilde{\tau} \times \tilde{\pi})$  extend to entire functions of  $s \in \mathbb{C}$ .
- (2)  $L(s, \tau \times \pi)$  and  $L(s, \tilde{\tau} \times \tilde{\pi})$  are bounded in vertical strips.
- (3) The functional equation  $L(s, \tau \times \pi) = \epsilon(s, \tau \times \pi) L(s, \tilde{\tau} \times \tilde{\pi})$  holds.

*Proof.* Twisting by  $\eta$  is necessary for conditions (1) and (2). Both (2) and (3) hold in wide generality.

Condition (2) follows from [GS, Cor. 4.5] and is valid for all  $\tau \in \mathcal{T}(N-1)$ , provided that one removes neighborhoods of the finite number of possible poles of the  $L$ -function. Condition (3) is a consequence of [Sh3, Thm. 7.7] and is valid for all  $\tau \in \mathcal{T}(N-1)$ .

Condition (1) follows from a more general result, [KS1, Prop. 2.1]. Note that this result rests on Assumption 1.1 of [KS1], sometimes called Assumption A [K1], on certain normalized intertwining operators being holomorphic and non-zero. Fortunately the assumption has been verified in our cases. The assumption requires two ingredients: the so-called “standard modules conjecture” and the “tempered  $L$ -functions conjecture”. Both of these have been verified in our cases in Wook Kim’s thesis [WKim]. For results proving various cases of this assumption we refer to [Sh3, CSh, MuSh, Mu, A, K3, Hei, KK]. Recently V. Heiermann and E. Opdam have proved the assumption in full generality in [HO].  $\square$

The key now is to relate the  $L$ -functions  $L(s, \tau \times \pi)$ , defined via the Langlands-Shahidi method, to the  $L$ -functions  $L(s, \Pi \times \tau)$  in the converse theorem. We note that, when we introduce our candidate for  $\Pi$ , for archimedean places and those non-archimedean places at which all data are unramified we know the equality of the local  $L$ -functions. However, we do not know this to be the case for ramified places. We get around this problem through stability of  $\gamma$ -factors, which basically makes the choice of local components of  $\Pi$  at the ramified places irrelevant as long as we can twist by highly ramified characters.

**3.3. Stability of  $\gamma$ -factors.** In this subsection let  $F$  denote a non-archimedean local field of characteristic zero. Let  $G = \mathrm{GSpin}^*(2n, F)$ , where the quasi-split non-split group is associates with a quadratic extension  $E/F$ .

Fix a nontrivial additive character  $\psi$  of  $F$ . Let  $\pi$  be an irreducible, admissible,  $\psi$ -generic representation of  $G$  and let  $\eta$  denote a continuous character of  $\mathrm{GL}(1, F)$ . Let  $\gamma(s, \eta \times \pi, \psi)$  be the associated  $\gamma$ -factor defined via the Langlands-Shahidi method [Sh3, Theorem 3.5]. We have

$$\gamma(s, \eta \times \pi, \psi) = \frac{\epsilon(s, \eta \times \pi, \psi) L(1-s, \eta^{-1} \times \tilde{\pi})}{L(s, \eta \times \pi)}.$$

**Proposition 3.7.** *Let  $\pi_1$  and  $\pi_2$  be two irreducible, admissible,  $\psi$ -generic representations of  $G$  having the same central characters. Then for a suitably highly ramified character  $\eta$  of  $\mathrm{GL}(1, F)$  we have*

$$\gamma(s, \eta \times \pi_1, \psi_v) = \gamma(s, \eta \times \pi_2, \psi_v).$$

*Proof.* This is a special case of a more general theorem which is the main result of [CPSS1]. We note that in our case one has to apply that theorem to the self-associate maximal Levi subgroup  $\mathrm{GL}(1) \times \mathrm{GSpin}^*(2n)$  in  $\mathrm{GSpin}^*(2n+2)$  which does satisfy the assumptions of that theorem.  $\square$

**3.4. The Candidate Transfer.** We construct a candidate global transfer  $\Pi = \otimes_v \Pi_v$  as a restricted tensor product of its local components  $\Pi_v$ , irreducible, admissible representations of  $\mathrm{GL}(2n, k_v)$ . There are three cases to consider: (i) archimedean  $v$ , (ii) non-archimedean unramified  $v$ , (iii) non-archimedean ramified  $v$ .

**3.4.1. The archimedean transfer.** If  $v$  is an archimedean place of  $k$ , then by the local Langlands correspondence [L2, B] the representation  $\pi_v$  is parametrized by an admissible homomorphism  $\phi_v$  and we choose  $\Pi_v$  to be the irreducible, admissible representation of  $\mathrm{GL}(2n, k_v)$  parametrized by  $\Phi_v$  as in the statement of Theorem 3.3. We then have

$$L(s, \pi_v) = L(s, \iota_v \circ \phi_v) = L(s, \Pi_v) \tag{3.8}$$

and

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(s, \iota_v \circ \phi_v, \psi_v) = \epsilon(s, \Pi_v, \psi_v), \tag{3.9}$$

where the middle factors are the local Artin-Weil  $L$ - and  $\epsilon$ -factors attached to representations of the Weil group as in [T]. The other  $L$ - and  $\epsilon$ -factors are defined via the Langlands-Shahidi

method which, in the archimedean case, are known to be the same as the Artin factors defined through the arithmetic Langlands classification [Sh1].

If  $\tau_v$  is an irreducible, admissible representation of  $\mathrm{GL}(m, k_v)$ , then it is parametrized by an admissible homomorphism  $\phi'_v : W_{k_v} \longrightarrow \mathrm{GL}(m, \mathbb{C})$  and the tensor product homomorphism  $(\iota_v \circ \phi_v) \otimes (\iota'_v \circ \phi'_v) : W_{k_v} \longrightarrow \mathrm{GL}(2mn, \mathbb{C})$  is another admissible homomorphism and we again have

$$L(s, \pi_v \times \tau_v) = L(s, (\iota_v \circ \phi_v) \otimes (\iota'_v \circ \phi'_v)) = L(s, \Pi_v \times \tau_v) \quad (3.10)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, (\iota_v \circ \phi_v) \otimes (\iota'_v \circ \phi'_v), \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v). \quad (3.11)$$

Here,  $\iota'_v$  is just the identity map on  $\mathrm{GL}(m, \mathbb{C})$ . Hence, we get the following matching of the twisted local  $L$ - and  $\epsilon$ -factors.

**Proposition 3.12.** *Let  $v$  be an archimedean place of  $k$  and let  $\pi_v$  be an irreducible, admissible, generic representation of  $\mathrm{GSpin}^*(2n, k_v)$ ,  $\Pi_v$  its local functorial transfer to  $\mathrm{GL}(2n, k_v)$ , and  $\tau_v$  an irreducible, admissible, generic representation of  $\mathrm{GL}(m, k_v)$ . Then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v), \quad L(s, \tilde{\pi}_v \times \tilde{\tau}_v) = L(s, \tilde{\Pi}_v \times \tilde{\tau}_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v).$$

□

**3.4.2. The non-archimedean unramified transfer.** If  $v$  is a non-archimedean place of  $k$  such that  $\pi_v$  as well as all  $K_w/k_v$ , for  $w|v$ , are unramified, then by the arithmetic Langlands classification or the Satake classification [B, Sat], the representation  $\pi_v$  is parametrized by an unramified admissible homomorphism  $\phi_v : W_{k_v} \longrightarrow {}^L\mathbf{G}_v^0$ . Again we take  $\Phi_v$  as in the statement of the theorem. It defines an irreducible, admissible, unramified representation  $\Pi_v$  of  $\mathrm{GL}(2n, k_v)$  [HT, H1].

Given that  $\pi_v$  is unramified its parameter  $\phi_v$  factors through an unramified homomorphism into the maximal torus  ${}^L\mathbf{T}_v \hookrightarrow {}^L\mathbf{G}_v$ . Then  $\Phi_v$  has its image in a torus of  $\mathrm{GL}(2n, \mathbb{C})$  which is split and  $\Pi_v$  is the corresponding unramified representation. Then we have

$$L(s, \pi_v) = L(s, \Pi_v) \quad (3.13)$$

and

$$\epsilon(s, \pi_v, \psi_v) = \epsilon(s, \Pi_v, \psi_v) \quad (3.14)$$

and the factors on either side of the above equations can be expressed as products of one-dimensional abelian Artin factors by multiplicativity of the local factors.

Let  $\tau_v$  be an irreducible, admissible, generic, unramified representation of  $\mathrm{GL}(m, k_v)$ . Again appealing to the general multiplicativity of local factors [JPSS, Sh3, Sh4] we have

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v) \quad (3.15)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v). \quad (3.16)$$

Hence, we again get the following matching of the twisted local  $L$ - and  $\epsilon$ -factors.

**Proposition 3.17.** *Let  $v$  be a non-archimedean place of  $k$  and let  $\pi_v$  be an irreducible, admissible, generic, unramified representation of  $\mathrm{GSpin}^*(2n, k_v)$ ,  $\Pi_v$  its local functorial transfer to  $\mathrm{GL}(2n, k_v)$ , and  $\tau_v$  an irreducible, admissible, generic representation of  $\mathrm{GL}(m, k_v)$ . Then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v), \quad L(s, \tilde{\pi}_v \times \tilde{\tau}_v) = L(s, \tilde{\Pi}_v \times \tilde{\tau}_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v).$$

□

We also make the local transfer in this case explicit. The analysis is similar to that of the quasi-split  $\mathrm{SO}(2n)$  carried out in [CPSS2, §7.2], which we refer to for more detail.

The unramified principal series representation  $\pi_v$  is given by an unramified character

$$\chi = (\chi_0, \chi_1, \dots, \chi_{n-1}, \chi_n^1)$$

of the maximal torus in  $\mathrm{GSpin}^*(2n, k_v)$ . Here,  $\chi_0, \chi_1, \dots, \chi_{n-1}$  are characters of  $k_v^\times$  with  $\chi_0$  being the central character of  $\pi_v$  and  $\chi_n^1$  is a character of  $K_w^1$ , elements of norm one in  $K_w$  embedded in  $\mathrm{GL}(2, k_v)$  as in [LL]. By Hilbert Theorem 90 we have  $K_w^\times/k_v^\times \cong K_w^1$ , which allows us to extend  $\chi_n^1$  to a character  $\tilde{\chi}_n$  of  $K_w^\times$ . The local transferred representation  $\Pi_v$  is then given by the character

$$\tilde{\chi} = (\chi_1, \dots, \chi_{n-1}, \tilde{\chi}_n, \chi_{n-1}^{-1}\chi_0, \dots, \chi_1^{-1}\chi_0)$$

of a (non-split) torus in  $\mathrm{GL}(2n, k_v)$ . (See [AS1, §6] for the split case.) To get principal series on  $\mathrm{GL}(2n, k_v)$  the character  $\tilde{\chi}_n$  must factor through the norm map. Write

$$\tilde{\chi}_n = \chi_n \circ N_{K_w/k_v}$$

with  $\chi_n$  a character of  $k_v^\times$  satisfying  $\chi_n^2 = \chi_0$ . We get the principal series representation

$$I(\chi_n \mu_{K_w/k_v}, \chi_n) = I(\chi_n \mu_{K_w/k_v}, \chi_n^{-1} \chi_0)$$

of  $\mathrm{GL}(2, k_v)$ , where  $\mu_{K_w/k_v}$  is the quadratic character of  $k_v^\times$  associated with the quadratic extension  $K_w/k_v$  by local class field theory. Hence,  $\pi_v$  transfers to the unramified principal series representation of  $\mathrm{GL}(2n, k_v)$  induced from the character

$$(\chi_1, \dots, \chi_{n-1}, \chi_n \mu_{K_w/k_v}, \chi_n^{-1} \chi_0, \chi_{n-1}^{-1} \chi_0, \dots, \chi_1^{-1} \chi_0).$$

It is now clear that the central character of  $\Pi_v$  is  $\chi_0^n \mu_{K_w/k_v}$  and its contragredient is isomorphic to  $\Pi_v \otimes \chi_0^{-1}$ . Therefore, we have proved the following.

**Proposition 3.18.** *Let  $v$  be a non-archimedean place of  $k$  and let  $\pi_v$  be an irreducible, admissible, generic, unramified representation of  $\mathrm{GSpin}^*(2n, k_v)$  with  $\Pi_v$  its local functorial transfer to  $\mathrm{GL}(2n, k_v)$  defined above. Then*

$$\omega_{\Pi_v} = \omega_{\pi_v}^n \mu_v$$

and

$$\Pi_v \cong \tilde{\Pi}_v \otimes \omega_{\pi_v}.$$

Here,  $\mu_v$  is a quadratic character of  $k_v^\times$  associated with  $\mathrm{GSpin}^*(2n, k_v)$ . □

**3.4.3. The non-archimedean ramified transfer.** For  $v$  a non-archimedean ramified place of  $k$  we take  $\Pi_v$  to be an arbitrary, irreducible, admissible representation of  $\mathrm{GL}(2n, k_v)$  whose central character satisfies

$$\omega_{\Pi_v} = \omega_{\pi_v}^n \mu_v,$$

where  $\mu_v$  is the quadratic character associated with the quadratic extension  $K_w/k_v$ .

We can no longer expect equality of  $L$ - and  $\epsilon$ -factors as in the previous cases. However, we do still get equality if we include a highly ramified character thanks to stability of  $\gamma$ -factors.

**Proposition 3.19.** *Let  $v$  be a non-archimedean ramified place of  $k$  and let  $\pi_v$  be an irreducible, admissible, generic representation of  $\mathrm{GSpin}^*(2n, k_v)$  and let  $\Pi_v$  be an irreducible, admissible representation of  $\mathrm{GL}(2n, k_v)$  as above. If  $\tau_v = \tau'_v \otimes \eta_v$  is an irreducible, admissible, generic representation of  $\mathrm{GL}(m, k_v)$  with  $\eta_v$  a sufficiently ramified character of  $\mathrm{GL}(1, k_v)$ , then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v), \quad L(s, \tilde{\pi}_v \times \tilde{\tau}_v) = L(s, \tilde{\Pi}_v \times \tilde{\tau}_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \epsilon(s, \Pi_v \times \tau_v, \psi_v).$$

*Proof.* The representation  $\tau'_v$  can be written as a full induced principal series

$$\tau_v = \mathrm{Ind}(\nu^{b_1} \otimes \cdots \otimes \nu^{b_m}) \otimes \eta_v = \mathrm{Ind}(\eta_v \nu^{b_1} \otimes \cdots \otimes \eta_v \nu^{b_m}),$$

where  $\nu(\cdot) = |\cdot|_v$ . By multiplicativity of the  $L$ - and  $\epsilon$ -factors we have

$$L(s, \pi_v \times \tau_v) = \prod_{i=1}^m L(s + b_i, \pi_v \times \eta_v)$$

and

$$\epsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^m L(s + b_i, \pi_v \times \eta_v, \psi_v).$$

Similarly,

$$L(s, \Pi_v \times \tau_v) = \prod_{i=1}^m L(s + b_i, \Pi_v \times \eta_v)$$

and

$$\epsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^m L(s + b_i, \Pi_v \times \eta_v, \psi_v).$$

This reduces the proof to the case of  $m = 1$ .

Next, note that because  $\eta_v$  is sufficiently ramified (depending on  $\pi_v$ ) the  $L$ -functions stabilize to one and we have

$$L(s, \pi_v \times \eta_v) \equiv 1$$

and

$$\epsilon(s, \pi_v \times \eta_v, \psi_v) = \gamma(s, \pi_v \times \eta_v, \psi_v).$$

On the other hand, by stability of gamma factors Proposition 3.7 we may replace  $\pi_v$  with another representation with the same central character. Hence, for  $n$  arbitrary characters  $\chi_1, \chi_2, \dots, \chi_n$ ,  $\chi_0 = \omega_{\pi_v}$  and the quadratic character  $\mu_v$  we have

$$\begin{aligned} \gamma(s, \pi_v \times \eta_v, \psi_v) &= \left( \prod_{i=1}^{n-1} \gamma(s, \eta_v \chi_i, \psi_v) \gamma(s, \eta_v \chi_i^{-1} \chi_0, \psi_v) \right) \\ &\quad \cdot \gamma(s, \eta_v \chi_n \mu_v, \psi_v) \gamma(s, \eta_v \chi_n^{-1} \chi_0, \psi_v). \end{aligned}$$

We refer to [AS1, §6] for more details in the split case. The calculations in the quasi-split case are similar, the only difference being the appearance of the quadratic character  $\mu_v$ .

We have similar relations also for the GL case. More precisely, because  $\omega_{\Pi_v} = \chi_0^n \mu_v$ , by [JS3, Proposition 2.2] we have

$$L(s, \Pi_v \times \eta_v) \equiv 1$$

and

$$\begin{aligned} \epsilon(s, \Pi_v \times \eta_v, \psi_v) &= \left( \prod_{i=1}^{n-1} \gamma(s, \eta_v \chi_i, \psi_v) \gamma(s, \eta_v \chi_i^{-1} \chi_0, \psi_v) \right) \\ &\cdot \gamma(s, \eta_v \chi_n \mu, \psi_v) \gamma(s, \eta_v \chi_n^{-1} \chi_0, \psi_v). \end{aligned}$$

Note that this is a special case of the multiplicativity of the local factors. This gives the equalities for the case of  $m = 1$  and hence completes the proof.  $\square$

**3.5. Proof of Theorem 3.3.** Let  $\omega$  denote the central character  $\omega_\pi$  of  $\pi$  and let  $S$  be as in the statement of Theorem 3.3. We let  $\Pi = \otimes_v \Pi_v$  with  $\Pi_v$  the candidates we constructed in 3.4.1–3.4.3. Also, let  $\mu = \otimes_v \mu_v$  be a quadratic idèle class character associated, by class field theory, with the quadratic extension  $K/k$ , where  $K$  is the field over which  $\mathrm{GSpin}^*(2n)$  is split.

Choose an idèle class character  $\eta$  of  $k$  which is sufficiently ramified at places  $v \in S$  so that the requirements of Propositions 3.6 and 3.7 are satisfied. We apply Theorem 3.5 to the representation  $\Pi$  and  $\mathcal{T}(S; \eta)$  with  $S$  and  $\eta$  as above.

By construction the central character  $\omega_\Pi$  of  $\Pi$  is equal to  $\omega^n \mu$ . Therefore, it is invariant under  $k^\times$ . Moreover, by (3.8) and (3.13) we have

$$L^S(s, \Pi) = \prod_{v \notin S} L(s, \Pi_v) = \prod_{v \notin S} L(s, \pi_v) = L^S(s, \pi).$$

This implies that  $L(s, \Pi) = \prod_v L(s, \Pi_v)$  is absolutely convergent in some right half plane. Furthermore, by Propositions 3.12, 3.17, and 3.19 we are free to check the remaining properties of being nice for the  $L$ -functions  $L(s, \Pi \times \tau)$  and the corresponding  $\epsilon$ -factors, for  $\tau \in \mathcal{T}(S; \eta)$ , instead for the  $L$ -functions  $L(s, \pi \times \tau)$  and its corresponding  $\epsilon$ -factors. Now the converse theorem can be applied thanks to Proposition 3.6 to conclude that there exists an automorphic representation of  $\mathrm{GL}(2n, \mathbb{A}_k)$  whose local components at  $v \notin S$  agree with those of  $\Pi$ . This automorphic representation is what we are calling  $\Pi$  in the statement of the theorem.

Finally, by Proposition 3.18, outside the finite set  $S \cup \{v : v|\infty\}$ , the central character of the automorphic representation  $\Pi$  agrees with the idèle class character  $\omega^n \mu$ , which implies that it is equal to  $\omega^n \mu$ . The same proposition also gives that  $\Pi_v \cong \tilde{\Pi}_v \otimes \omega_{\pi_v}$  for  $v \notin S \cup \{v : v|\infty\}$ . This completes the proof.  $\square$

#### 4. THE INTEGRAL FOR $\mathrm{GL}(m) \times \mathbf{G}(n)$

In this section we develop the analogue of the theory of Gelbart, Ginzburg, Piatetski-Shapiro, Rallis and Soudry in our situation.

Let  $n \geq 0$  and denote by  $\mathbf{G} = \mathbf{G}(n)$  either  $\mathrm{GSpin}(2n+1)$  or a quasi-split  $\mathrm{GSpin}(2n)$ . We write down and analyze a zeta integral which will give the  $L$ -functions for  $\mathrm{GL}(m) \times \mathbf{G}(n)$  for  $m \leq n$ . For  $m = n$  the construction and analysis of this zeta integral when  $\mathbf{G}$  is a special orthogonal group is due to Gelbart and Piatetski-Shapiro [GPS] using their Methods A and B. (They also cover  $\mathbf{G}$  symplectic.) Ginzburg then extended their work to the case of  $m < n$  and  $\mathbf{G}$  a special orthogonal group in [G]. We would like to carry out the same construction and analysis for  $m \leq n$  and  $\mathbf{G}$  a general spin group. We first need to define a certain unipotent subgroup of  $\mathbf{G}$  just as in [G]. For  $m < n$  we use the embeddings we described in Section 2.2. We point out that this subgroup will be trivial if  $m = n$  and Ginzburg's integral reduces to that of [GPS].



**4.1. Unipotent subgroups.** We define a subgroup  $\mathbf{N}$  of  $\mathbf{G}(n)$  generated by the subgroups  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{U}_{2(n-m)+1}$  in the odd case or  $\mathbf{U}_{2(n-m)}$  in the even case, as follows.

Each of the three subgroups is generated by a family of root groups. Let

$$\mathbf{U}_{2(n-m)+1} = \left\langle \mathbf{U}_\alpha \mid \alpha = \begin{cases} e_i \pm e_j, & m+1 \leq i < j \leq n \\ e_\ell, & m+1 \leq \ell \leq n \end{cases} \right\rangle$$

in the odd case or

$$\mathbf{U}_{2(n-m)} = \langle \mathbf{U}_\alpha \mid \alpha = e_i \pm e_j, \quad m+1 \leq i < j \leq n \rangle$$

in the even case, i.e.,  $\alpha$  is a positive root which can be written as a linear combination of the simple roots of  $\mathbf{G}(n)$  not involving the first  $m$ . Clearly  $\mathbf{U}_{2(n-m)+1}$  embeds naturally in the maximal unipotent subgroup  $\mathbf{U} = \mathbf{U}_{2n+1}$  of  $\mathrm{GSpin}(2n+1)$  and  $\mathbf{U}_{2(n-m)}$  embeds in  $\mathbf{U} = \mathbf{U}_{2n}$ , the maximal unipotent subgroup of  $\mathrm{GSpin}(2n)$ .

Moreover, let

$$\mathbf{X} = \left\langle \mathbf{U}_{-(e_i - e_j)} \mid \begin{array}{l} 1 \leq i \leq m, \\ m+1 \leq j \leq n \end{array} \right\rangle,$$

and

$$\mathbf{Y} = \left\langle \mathbf{U}_{(e_i + e_j)} \mid \begin{array}{l} 1 \leq i \leq m, \\ m+1 \leq j \leq n \end{array} \right\rangle$$

in either case.

**4.2. The Zeta Integral.** Let  $\pi$  be a generic, cuspidal, automorphic representation of  $\mathbf{G}(n, \mathbb{A})$  and let  $\tau$  be a cuspidal, automorphic representation of  $\mathrm{GL}(m, \mathbb{A})$ . Denote by  $U = \mathbf{U}(\mathbb{A})$  the maximal unipotent subgroup of  $\mathbf{G}(\mathbb{A})$  generated by  $U_\alpha$  for  $\alpha \in R^+$ . Every  $u \in U$  can be written uniquely as

$$u = \prod_{\alpha \in R^+} u_\alpha(x_\alpha). \quad (4.1)$$

Let  $\psi$  be a non-degenerate (additive) character of  $k \backslash \mathbb{A}$  and extend it to a non-degenerate character, again denoted by  $\psi$ , of  $U$  via

$$\psi(u) = \psi \left( \sum_{\alpha \in \Delta} x_\alpha \right). \quad (4.2)$$

We assume that  $\pi$  is  $\psi$ -generic, i.e., the space  $\mathcal{W}(\pi, \psi)$  (Whittaker model) of functions

$$W_\phi(g) = \int_{\mathbf{U}(k) \backslash \mathbf{U}(\mathbb{A})} \phi(ug) \psi(u) du, \quad \phi \in \pi, g \in \mathbf{G}(\mathbb{A}), \quad (4.3)$$

is non-zero. Recall that cuspidal, automorphic representations of  $\mathrm{GL}(m, \mathbb{A})$ , such as  $\tau$ , are automatically generic.

The group  $\mathbf{M}_m = \mathrm{GL}(m) \times \mathrm{GL}(1)$  sits inside either of  $\mathbf{H} = \mathbf{H}(m) = \mathrm{GSpin}(2m+1)$  or  $\mathrm{GSpin}(2m)$  as the Levi component of the standard Siegel parabolic  $\mathbf{P}_m = \mathbf{M}_m \mathbf{N}_m$ . This parabolic corresponds to the subset

$$\theta = \Delta - \{\alpha_m\}$$

of  $\Delta$ . We choose  $\mathbf{H}$  of the type opposite to  $\mathbf{G}$ . In other words, if  $\mathbf{G}$  is of type  $B_n$ , then we take  $\mathbf{H}$  of type  $D_m$  and vice versa. We also denote the maximal unipotent subgroup of  $\mathbf{M}_m$  by  $\mathbf{Z}_m$ . Therefore,  $\mathbf{Z}_m \mathbf{N}_m = \mathbf{U}_\mathbf{H}$ , the maximal unipotent subgroup of  $\mathbf{H} = \mathbf{H}(m)$ .

Let  $\omega$  be an idèle class character of  $\mathrm{GL}(1, \mathbb{A})$ . Let

$$\tau'_s = \tau |\det|^{s-1/2} \otimes \omega \quad (4.4)$$

be a representation of  $\mathbf{M}_m(\mathbb{A})$ , and extend it trivially across  $\mathbf{N}_m(\mathbb{A})$  to obtain a representation of  $\mathbf{P}_m(\mathbb{A})$ . Consider the normalized induced representation

$$\mathrm{Ind}_{\mathbf{P}_m(\mathbb{A})}^{\mathbf{H}(\mathbb{A})}(\tau'_s). \quad (4.5)$$

For  $f_{\tau'_s}$  in this induced representation construct the Eisenstein series

$$E(h, f_{\tau'_s}) = \sum_{\gamma \in \mathbf{P}_m(k) \backslash \mathbf{H}(k)} f_{\tau'_s}(\gamma h), \quad h \in \mathbf{H}(\mathbb{A}). \quad (4.6)$$

Moreover, we define the “quasi-Whittaker functions”  $W_{f_{\tau'_s}}$  as follows. Recalling that  $\mathbf{Z}_m$  is the maximal unipotent subgroup of the Siegel Levi  $\mathbf{M}_m = \mathrm{GL}(1) \times \mathrm{GL}(m)$  in  $\mathbf{H}$ , regard  $\psi$  as a character of  $\mathbf{Z}_m(\mathbb{A})$  through  $\psi(z) = \psi(\sum z_{i,i+1})$ . With  $f_{\tau'_s}$  as before we define

$$W_{f_{\tau'_s}}(h) = \int_{\mathbf{Z}_m(k) \backslash \mathbf{Z}_m(\mathbb{A})} f(zh) \psi(z) dz, \quad h \in \mathbf{H}(\mathbb{A}). \quad (4.7)$$

We are now prepared to define the zeta integral as follows. Let  $\phi$  be in the space of  $\pi$  and let  $f_{\tau'_s}$  be as above. Recall that  $\mathbf{X}$ ,  $\mathbf{Y}$  and  $\mathbf{U}_\ell$  with  $\ell = 2(n-m)+1$  or  $\ell = 2(n-m)$  are unipotent subgroups of  $\mathbf{G}(n)$  and we have the embedding  $\mathbf{i} : \mathbf{H} \hookrightarrow \mathbf{G}(n)$ , where  $\mathbf{H} = \mathrm{GSpin}(2m)$  or  $\mathrm{GSpin}(2m+1)$ , respectively. Define

$$\begin{aligned} I(\phi, f_{\tau'_s}) &= \int_{\mathbf{X}(k) \backslash \mathbf{X}(\mathbb{A})} \int_{\mathbf{Y}(k) \backslash \mathbf{Y}(\mathbb{A})} \int_{\mathbf{U}_\ell(k) \backslash \mathbf{U}_\ell(\mathbb{A})} \int_{\mathbf{H}(k) \backslash \mathbf{H}(\mathbb{A})} \\ &\quad \phi[u_\ell \cdot y \cdot x \cdot \mathbf{i}(h)] \cdot \psi(u_\ell) \cdot E(h, f_{\tau'_s}) dh du_\ell dy dx. \end{aligned} \quad (4.8)$$

We should remark here that (4.8) defines a Rankin-Selberg type integral as follows. Recalling that  $\mathbf{N}$  denotes the subgroup generated by  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{U}_\ell$  and setting

$$\phi_N(h) = \int_{\mathbf{N}(k) \backslash \mathbf{N}(\mathbb{A})} \phi(n \cdot \mathbf{i}(h)) \psi(n) dn, \quad h \in \mathbf{H}(\mathbb{A}),$$

we have

$$I(\phi, f_{\tau'_s}) = \int_{\mathbf{H}(k) \backslash \mathbf{H}(\mathbb{A})} \phi_N(h) E(h, f_{\tau'_s}) dh.$$

**4.3. The Basic Identity.** We now state and prove the basic identity for the zeta integrals we just introduced. We start with a lemma. This lemma is an analogue of Ginzburg’s lemma in [G, p. 172]. For a special more explicit statement of this lemma see [G, p. 168]. The proof of the lemma also carries over, more or less word for word, from that of Ginzburg’s lemma in [G, p. 172]. We remark that we are using slightly different notation from Ginzburg’s paper. For example, we are making a distinction between  $\mathbf{H}$  and the image of its embedding in  $\mathbf{G}$  through the use of the map  $\mathbf{i}$ . Also we are using the notation  $\mathbf{N}_m$  for the unipotent radical of the parabolic  $\mathbf{P}_m$  reserving  $\mathbf{M}_m$  for its Levi component while Ginzburg’s paper uses  $M_k$  (with his  $k$  being our  $m$ ) for the unipotent radical.

**Lemma 4.9.** *With notation as in the previous section we have*

$$\begin{aligned} \int_{\mathbf{X}(\mathbb{A})} \sum_{\gamma \in \mathbf{Z}_m(k) \backslash \mathbf{M}_m(k)} W_\phi(\mathbf{i}(\gamma) xg) dx &= \int_{\mathbf{N}_m(k) \backslash \mathbf{N}_m(\mathbb{A})} \int_{\mathbf{X}(k) \backslash \mathbf{X}(\mathbb{A})} \int_{\mathbf{Y}(k) \backslash \mathbf{Y}(\mathbb{A})} \int_{\mathbf{U}_\ell(k) \backslash \mathbf{U}_\ell(\mathbb{A})} \\ &\quad \phi[u_\ell \cdot y \cdot x \cdot n \cdot g] \psi(u_\ell) du_\ell dy dx dn, \quad \forall g \in \mathbf{G}(\mathbb{A}). \end{aligned}$$

We now state the basic identity involving the zeta integrals, the main result of this section.

**Theorem 4.10.** (a)  $I(\phi, f_{\tau'_s})$  converges for all  $s$ .  
 (b)

$$I(\phi, f_{\tau'_s}) = \int_{\mathbf{U}_{\mathbf{H}}(\mathbb{A}) \backslash \mathbf{H}(\mathbb{A})} \int_{\mathbf{X}(\mathbb{A})} W_{\phi}(\mathbf{i}(h)x) W_{f_{\tau'_s}}(h) dx dh.$$

(c)  $I(\phi, f_{\tau'_s})$  has a meromorphic continuation and satisfies the functional equation.

$$I(\phi, f_{\tau'_s}) = I\left(\phi, M(s)f_{\tau'_{1-s}}\right).$$

*Proof.* The proof is similar to that of [G, Theorem A] which we closely follow.

To prove part (a) consider the integral on the right hand side of (4.8). First, we would like to replace the integration over  $\mathbf{H}(k) \backslash \mathbf{H}(\mathbb{A})$  by a Siegel set. Recall that a Siegel set in  $\mathbf{H}(\mathbb{A})$  is a set of the form

$$S_H = U_H^0 A_c K_H,$$

where  $U_H^0$  is a relatively compact subset of the maximal unipotent subgroup  $\mathbf{U}_{\mathbf{H}}(\mathbb{A})$  in  $\mathbf{H}(\mathbb{A})$ ,  $K_H$  is a maximal compact subgroup in  $\mathbf{H}(\mathbb{A})$ , and  $A_c$  consists of elements  $a$  in the maximal torus of  $\mathbf{H}(\mathbb{A})$  satisfying  $|\alpha(a)| \geq c$  for all simple roots  $\alpha$  of  $\mathbf{H}$ . By reduction theory we know that  $\mathbf{H}(\mathbb{A}) = \mathbf{H}(k)S_H$ .

Choose a Siegel set  $S_H = U_H^0 A_{c_1} K_H$  in  $\mathbf{H}(\mathbb{A})$  as above so that (4.8) can be written as

$$\begin{aligned} I(\phi, f_{\tau'_s}) &= \int_{\mathbf{X}(k) \backslash \mathbf{X}(\mathbb{A})} \int_{\mathbf{Y}(k) \backslash \mathbf{Y}(\mathbb{A})} \int_{\mathbf{U}_{\ell}(k) \backslash \mathbf{U}_{\ell}(\mathbb{A})} \int_{U_H^0} \int_{A_{c_1}} \int_{K_H} \\ &\quad \phi[u_{\ell} y x \mathbf{i}(u') \mathbf{i}(a) \mathbf{i}(k)] \cdot \psi(u_{\ell}) \cdot E(u' a k, f_{\tau'_s}) du' da dk du_{\ell} dy dx. \end{aligned} \quad (4.11)$$

Note that we have  $x \mathbf{i}(u') = \mathbf{i}(u') x$ . This follows from the definitions of  $\mathbf{X}$  and of the embedding  $\mathbf{i}$  because the root groups defining  $X$  and  $\mathbf{i}(u')$  commute, a fact that follows from [AS1, (21)], for example. Hence, we are allowed to change the order of  $x \mathbf{i}(u')$  to  $\mathbf{i}(u') x$  in the integral.

Next, we choose a Siegel set  $S_G = U_G^0 R_c K_G$  in  $\mathbf{G}(\mathbb{A})$  in a similar way. Write

$$x \mathbf{i}(a) = u'' r k, \quad u'' \in U_G^0, r \in R_c, k \in K_G. \quad (4.12)$$

Then the integral (4.11) becomes

$$I(\phi, f_{\tau'_s}) = \int_{U'} \int_{R_c} \int_{K_G} \phi(u' r k) \cdot \psi(u') \cdot E(u' r k, f_{\tau'_s}) du' dr dk, \quad (4.13)$$

where  $U'$  is some unipotent set in  $\mathbf{G}(\mathbb{A})$ .

Recall that  $\phi$  is a cusp form and hence *rapidly decreasing*. This means, in particular, that for any  $N \in \mathbb{Z}$  there exists a constant  $C_{\phi, N}$  such that

$$|\phi(u' r k)| \leq C_{\phi, N} |\alpha(r)|^N, \quad (4.14)$$

for all simple roots  $\alpha$  of  $\mathbf{G}$ . Moreover, the Eisenstein series  $E(u' r k, f_{\tau'_s})$  is *slowly increasing*. This means, in particular, that (4.14) holds, with  $\phi$  replaced by the Eisenstein series, for some  $N$ . We will use these facts to bound the integral on the right hand side of (4.13). Notice that we do not know  $r$  explicitly and, in particular, we do not know that  $|\alpha_i(r)| \rightarrow \infty$  for some  $i$ . Instead, we have to find some other way to bound the right hand side of (4.13).

Consider the equality (4.12). Writing

$$a = e_0^*(t_0) e_1^*(t_1) \dots e_m^*(t_m)$$

use Lemma 2.25 to see that

$$\mathbf{i}(a) = \begin{cases} f_0^*(t_0)f_1^*(t_1) \cdots f_{m-1}^*(t_{m-1})f_m^*(t_m) & \text{if } \mathbf{H} \text{ is even and } \mathbf{G} \text{ is odd,} \\ f_0^*(-t_0)f_1^*(t_1) \cdots f_{m-1}^*(t_{m-1})f_m^*(-t_m)f_n^*(-1) & \text{if } \mathbf{H} \text{ is odd and } \mathbf{G} \text{ is even.} \end{cases}$$

Also, write

$$r = f_0^*(r_0)f_1^*(t_1) \cdots f_n^*(t_n).$$

Now apply the character  $D = f_1 + \cdots + f_n$ , which lies in the character lattice, to both sides of (4.12). (This character amounts to the determinant on the  $\mathrm{GL}(n)$  part of the Siegel Levi.) Notice that the elements on both sides are indeed inside the Siegel Levi in  $\mathbf{G}$ . We conclude that

$$|t_1 t_2 \cdots t_m| = |r_1 r_2 \cdots r_n|.$$

By definition of a Siegel set, on  $R_c$ , we have

$$|r_1| \geq c|r_2|, \dots, |r_{n-1}| \geq c|r_n|$$

in both even and odd cases. This implies that

$$\begin{aligned} |t_1 t_2 \cdots t_m| &= |r_1 r_2 \cdots r_n| \\ &\leq |r_1| \cdot c^{-1} |r_1| \cdot c^{-2} |r_1| \cdots c^{n-1} |r_1| \\ &= c^{-n(n-1)/2} |\alpha_1(r) \cdots \alpha_n(r)|^n. \end{aligned}$$

Since  $|t_1 t_2 \cdots t_m| \rightarrow \infty$  we have that  $|\alpha_1(r) \cdots \alpha_n(r)| \rightarrow \infty$ . Therefore, we conclude from (4.13) that

$$|I(\phi, f_{\tau'_s})| \leq \int_{R_c} |\alpha_1(r) \cdots \alpha_n(r)|^{-N} p_s(|r_1 \cdots r_n|) dr, \quad \forall N \in \mathbb{Z},$$

where  $p_s(|r_1 \cdots r_n|)$  is a polynomial in  $|r_1 \cdots r_n|$ . Since  $|\alpha_1(r) \cdots \alpha_n(r)| \rightarrow \infty$ , the last integral converges for any fixed  $s$  if we take  $N$  to be large enough. This proves part (a).

In order to prove part (b) start with the definition (4.8) of  $I(\phi, f_{\tau'_s})$  and unfold the Eisenstein series as in (4.6) to get

$$\begin{aligned} I(\phi, f_{\tau'_s}) &= \int_{\mathbf{X}(k) \backslash \mathbf{X}(\mathbb{A})} \int_{\mathbf{Y}(k) \backslash \mathbf{Y}(\mathbb{A})} \int_{\mathbf{U}_\ell(k) \backslash \mathbf{U}_\ell(\mathbb{A})} \int_{\mathbf{H}(k) \backslash \mathbf{H}(\mathbb{A})} \\ &\quad \phi[u_\ell y x n \mathbf{i}(h)] \cdot \psi(u_\ell) \cdot \sum_{\gamma \in \mathbf{P}_m(k) \backslash \mathbf{H}(k)} f_{\tau'_s}(\gamma h) dh du_\ell dy dx \\ &= \int_{\mathbf{X}(k) \backslash \mathbf{X}(\mathbb{A})} \int_{\mathbf{Y}(k) \backslash \mathbf{Y}(\mathbb{A})} \int_{\mathbf{U}_\ell(k) \backslash \mathbf{U}_\ell(\mathbb{A})} \int_{\mathbf{P}_m(k) \backslash \mathbf{H}(\mathbb{A})} \\ &\quad \phi[u_\ell y x n \mathbf{i}(h)] \cdot \psi(u_\ell) \cdot f_{\tau'_s}(\gamma h) dh du_\ell dy dx. \end{aligned}$$

Writing

$$\int_{\mathbf{P}_m(k) \backslash \mathbf{H}(\mathbb{A})} = \int_{\mathbf{M}_m(k) \backslash \mathbf{N}_m(\mathbb{A}) \backslash \mathbf{H}(\mathbb{A})} \int_{\mathbf{N}_m(k) \backslash \mathbf{N}_m(\mathbb{A})}$$

and changing the order of integration we get

$$\begin{aligned} I(\phi, f_{\tau'_s}) &= \int_{\mathbf{M}_m(k) \backslash \mathbf{N}_m(\mathbb{A}) \backslash \mathbf{H}(\mathbb{A})} \int_{\mathbf{N}_m(k) \backslash \mathbf{N}_m(\mathbb{A})} \int_{\mathbf{X}(k) \backslash \mathbf{X}(\mathbb{A})} \int_{\mathbf{Y}(k) \backslash \mathbf{Y}(\mathbb{A})} \int_{\mathbf{U}_\ell(k) \backslash \mathbf{U}_\ell(\mathbb{A})} \\ &\quad \phi[u_\ell y x n \mathbf{i}(h)] \cdot \psi(u_\ell) \cdot f_{\tau'_s}(nh) du_\ell dy dx dn dh. \end{aligned}$$

Now we have  $f_{\tau'_s}(nh) = f_{\tau'_s}(h)$ . Applying Lemma 4.9 we get

$$I(\phi, f_{\tau'_s}) = \int_{\mathbf{M}_m(k)\mathbf{N}_m(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \int_{\mathbf{X}(\mathbb{A})} \sum_{\gamma \in \mathbf{Z}_m(k)\backslash\mathbf{M}_m(k)} W_\phi(\mathbf{i}(\gamma) \cdot x \cdot \mathbf{i}(h)) f_{\tau'_s}(h) dx dh.$$

Next, because  $\mathbf{X}$  is normalized by  $\mathbf{i}(\gamma)$ , after making a change of variables if necessary, we may change  $\mathbf{i}(\gamma)x$  to  $x\mathbf{i}(\gamma)$ . Moreover, because  $f_{\tau'_s}$  is in a space induced from cusp forms and  $\gamma \in \mathbf{M}_m(k)$  we have  $f_{\tau'_s}(\gamma\mathbf{i}(h)) = f_{\tau'_s}(\mathbf{i}(h))$ . Hence,

$$I(\phi, f_{\tau'_s}) = \int_{\mathbf{Z}_m(k)\mathbf{N}_m(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \int_{\mathbf{X}(\mathbb{A})} W_\phi(x \cdot \mathbf{i}(h)) f_{\tau'_s}(h) dx dh.$$

Furthermore, we may change the order of integration from  $x\mathbf{i}(h)$  to  $\mathbf{i}(h)x$  since  $x\mathbf{i}(h) = u'\mathbf{i}(h)x'$  with  $x'$  in the maximal unipotent subgroup of  $\mathbf{G}(\mathbb{A})$  only involving non-simple roots, hence  $\psi(u') = 1$ . Again writing

$$\int_{\mathbf{Z}_m(k)\mathbf{N}_m(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} = \int_{\mathbf{Z}_m(\mathbb{A})\mathbf{N}_m(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \int_{\mathbf{Z}_m(k)\backslash\mathbf{Z}_m(\mathbb{A})}$$

we get

$$I(\phi, f_{\tau'_s}) = \int_{\mathbf{Z}_m(\mathbb{A})\mathbf{N}_m(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \int_{\mathbf{X}(\mathbb{A})} W_\phi(\mathbf{i}(h)x) \left[ \int_{\mathbf{Z}_m(k)\backslash\mathbf{Z}_m(\mathbb{A})} \psi(z) f_{\tau'_s}(zh) dz \right] dh dx.$$

Using (4.7) we finally get

$$I(\phi, f_{\tau'_s}) = \int_{\mathbf{Z}_m(\mathbb{A})\mathbf{N}_m(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})} \int_{\mathbf{X}(\mathbb{A})} W_\phi(\mathbf{i}(h)x) W_{f_{\tau'_s}}(h) dx dh.$$

This is part (b). Part (c) follows from the definition (4.8) and the properties of the meromorphic continuation and the functional equation of the Eisenstein series (4.6).  $\square$

As a consequence of Theorem 4.10 we have the following Euler product expansion. With appropriate choices as in [GPS, pp. 93–94] we have a factorization  $W_\phi(g) = \prod_v W_v(g_v)$  and  $W_{f_{\tau'_s}}(g) = \prod_v W_{f_{\tau'_s}, v}(g_v)$  such that

$$I(\phi, f_{\tau'_s}) = \prod_v \xi(W_v, W_{f_{\tau'_s}, v}), \quad (4.15)$$

where

$$\xi(W_v, W_{f_{\tau'_s}, v}) = \int_{\mathbf{U}_\ell(k_v)\backslash\mathbf{H}(k_v)} \int_{\mathbf{X}(k_v)} W_v(g_v x_v) W_{f_{\tau'_s}, v}(g_v) dx dg. \quad (4.16)$$

**4.4. The Unramified Computations.** In this section we compute the zeta integral at the unramified places. This local analysis allows us to connect our integrals on the right hand side of (4.16) to local  $L$ -functions.

Let  $\pi = \otimes_v \pi_v$  and  $\tau'_s = \otimes_v \tau'_{s, v}$  be as before. Recall that  $\tau'_s$  is as in (4.4) in terms of a representation  $\tau$  of  $\mathrm{GL}(m, \mathbb{A})$  and an idèle class character  $\omega$  of  $\mathrm{GL}(1, \mathbb{A})$ .

**Theorem 4.17.** *Let  $v$  be a non-archimedean place of  $k$  such that  $\pi_v$  and  $\tau'_{s,v}$  are unramified. If  $W_{\pi_v}^0$  and  $W_{\tau'_{s,v}}^0$  are class one functions for the local, unramified representations  $\pi_v$  and  $\tau'_{s,v}$  respectively, then*

$$\int_{\mathbf{X}(k_v)} \int_{\mathbf{U}_\ell(k_v) \backslash \mathbf{H}(k_v)} W_{\pi_v}^0(gx) W_{\tau'_{s,v}}^0(g) dg dx = \begin{cases} \frac{L(s, \pi_v \times \tau_v)}{L(2s, \tau_v, \wedge^2 \otimes \omega^{-1})}, & \text{if } \mathbf{G} = \mathrm{GSpin}(2n+1), \\ \frac{L(s, \pi_v \times \tau_v)}{L(2s, \tau_v, \mathrm{Sym}^2 \otimes \omega^{-1})}, & \text{if } \mathbf{G} = \mathrm{GSpin}(2n). \end{cases}$$

*Proof.* The proof uses a decreasing induction on the GL rank and is completely similar to the proof of [G, Theorem B]. The starting step of the induction, as in Ginzburg's theorem, is similar to the case of  $\mathrm{GL}(n) \times \mathrm{SO}(2n+1)$  in [GPS]. One uses the Casselman-Shalika formula [CS] for the calculation of the Whittaker functions.  $\square$

**4.5. Global Zeta Integral and Partial  $L$ -functions.** We now state the major consequence of the above discussions in the global setting we need. We use the notation of the earlier sections.

**Theorem 4.18.** *Let  $\pi$  be a unitary, cuspidal, globally generic, automorphic representation of  $\mathbf{G}(n, \mathbb{A})$  and let  $\tau$  be a unitary, cuspidal representation of  $\mathrm{GL}(m, \mathbb{A})$ . Assume that  $m \leq n$ . Moreover, fix an idèle class character  $\omega$  of  $\mathrm{GL}(1, \mathbb{A})$ . For a sufficiently large finite set of places  $S$ , including all the archimedean places and the places where ramification occurs, we have*

$$I(\phi, f_{\tau'_s}) = \begin{cases} \frac{L^S(s, \pi \times \tau)}{L^S(2s, \tau, \wedge^2 \otimes \omega^{-1})} \cdot R(s), & \text{if } \mathbf{G} = \mathrm{GSpin}(2n+1), \\ \frac{L^S(s, \pi \times \tau)}{L^S(2s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})} \cdot R(s), & \text{if } \mathbf{G} = \mathrm{GSpin}(2n), \end{cases} \quad (4.19)$$

where  $R(s)$  is a meromorphic function, which can be made holomorphic and nonzero in a neighborhood of any given  $s = s_0$  for an appropriate choice of  $f$ .

*Proof.* The theorem follows from Theorem 4.17 if we set  $R(s)$  to be equal to the product of the local zeta integrals (4.16) over  $v \notin S$ . The fact that  $R(s)$  is meromorphic is clear. To show that it can be made holomorphic in the neighborhood of any point  $s = s_0$  the argument is completely similar to the one for the case of  $\mathrm{GL}(m) \times \mathrm{SO}(2n+1)$  in [Sou1, §§6-7].  $\square$

## 5. THE TRANSFERRED REPRESENTATION

In this section  $k$  will continue to denote a number field and  $\mathbb{A} = \mathbb{A}_k$  will denote its ring of adèles. Let  $\mathbf{G} = \mathbf{G}(n)$  denote  $\mathrm{GSpin}(2n+1)$ , the split  $\mathrm{GSpin}(2n)$ , or the quasi-split  $\mathrm{GSpin}^*(2n)$  associated with a quadratic extension  $K/k$  of number fields. We will refer to the case of  $\mathbf{G}(n) = \mathrm{GSpin}(2n+1)$  as the odd case and the remaining cases as the even case.

**5.1. The Global Transfer.** Let  $\pi$  be a irreducible, generic, unitary, cuspidal, automorphic representation of  $\mathbf{G}(\mathbb{A})$ . Let  $\Pi$  be a transfer of  $\pi$  to  $\mathrm{GL}(2n, \mathbb{A}_k)$  as in [AS1, Theorem 1.1] and Theorem 3.3. By the classification of automorphic representations of general linear groups [JS1, JS2] we know that  $\Pi$  is a constituent of some automorphic representation

$$\Sigma = \mathrm{Ind}(|\det|^{r_1} \sigma_1 \otimes \cdots \otimes |\det|^{r_t} \sigma_t) \quad (5.1)$$

with  $\sigma_i$  a unitary, cuspidal, automorphic representation of  $\mathrm{GL}(n_i, \mathbb{A})$ ,  $r_i \in \mathbb{R}$ , and  $n_1 + n_2 + \cdots + n_t = 2n$ .

Let  $\omega = \omega_\pi$  denote the central character of  $\pi$ . Then  $\omega$  is a unitary idèle class character of  $k$  and we have shown that  $\Pi$  is nearly equivalent to  $\tilde{\Pi} \otimes \omega$ .

Our first goal in this section is to prove the fact that all the exponents  $r_i = 0$  in (5.1). In order to do so, we follow the method of Gelbart, Ginzburg, Piatetski-Shapiro, Rallis and Soudry as explained for classical groups in [Sou2, §1]. We explained in Section 4 how to generalize parts of this theory to the cases of odd and even GSpin groups.

We start with a lemma about twisted exterior and symmetric (partial)  $L$ -functions. For its proof we need a result on holomorphy of twisted  $L$ -function in the half plane  $\Re(s) > 1$ . This result and much more are the subject of two works currently being completed, one by Dustin Belt in his thesis at Purdue University, and the other by Suichiro Takeda which has appeared as a preprint.

**Proposition 5.2.** ([Bl] and [Tk]) *Let  $\chi$  be an arbitrary idèle class character and let  $\tau$  be a unitary, cuspidal, automorphic representation of  $\mathrm{GL}(m, \mathbb{A})$ . Let  $S$  be a finite set of places of  $k$  containing all the archimedean places and all the non-archimedean places at which  $\pi$  ramifies. Then the partial twisted  $L$ -functions  $L^S(s, \pi, \wedge^2 \otimes \chi)$  and  $L^S(s, \pi, \mathrm{Sym}^2 \otimes \chi)$  are holomorphic in  $\Re(s) > 1$ .*

We remark that Jacquet and Shalika proved that  $L^S(s, \pi, \wedge^2 \otimes \chi)$  has a meromorphic continuation to a half plane  $\Re(s) > 1 - a$  with  $a > 0$  depending on the representation [JS4, §8, Theorem 1]. Proposition 5.2 in the case of  $\wedge^2 \otimes \omega$  can also be dug out of their work. However, D. Belt's results show this for all  $s$ , with possible poles at  $s = 0, 1$ .

As far as we know, an analogue of Jacquet-Shalika's result for twisted symmetric square was not available. For  $m = 3$  it follows from results of W. Banks [Bnk] following the untwisted ( $\chi = 1$ ) results of Bump and Ginzburg [BG]. S. Takeda's results build on this line of work.

**Lemma 5.3.** *Let  $m$  be a positive integer and let  $\tau$  be an irreducible, unitary, cuspidal, automorphic representation of  $\mathrm{GL}(m, \mathbb{A})$ . Let  $\omega$  be an idèle class character and let  $s \in \mathbb{C}$ . Let  $S$  be a finite set of places of  $k$  including all the archimedean ones such the data is unramified outside  $S$ .*

- (a) *Both  $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$  and  $L^S(s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})$  are holomorphic and non-vanishing for  $\Re(s) > 1$ .*
- (b) *If either of the above  $L$ -functions has a pole at  $s = 1$ , then  $\tau \cong \tilde{\tau} \otimes \omega$ .*

*Proof.* We have

$$L^S(s, \tau \otimes (\tau \otimes \omega^{-1})) = L^S(s, \tau, \wedge^2 \otimes \omega^{-1}) L^S(s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1}).$$

The left hand side is holomorphic and non-vanishing for  $\Re(s) > 1$  by [JS2, Proposition (3.6)]. Moreover, by Proposition 5.2 both of the  $L$ -functions on the right hand side are holomorphic for  $\Re(s) > 1$ . Therefore, both are non-vanishing there, as well. This is part (a).

On the other hand, by [Sh5, Theorem 1.1] both  $L$ -functions on the right hand side are non-vanishing on  $\Re(s) = 1$ . If one has a pole at  $s = 1$ , then the left hand side must have a pole at  $s = 1$ . Again by [JS2, Proposition (3.6)] the two representations  $\tau$  and  $\tau \otimes \omega^{-1}$  must be contragredient of each other, i.e.,  $\tau \cong \tilde{\tau} \otimes \omega$ . This is part (b).  $\square$

**Proposition 5.4.** *Let  $\tau$  be an irreducible, unitary, cuspidal representation of  $\mathrm{GL}(m, \mathbb{A})$  and let  $\omega$  be an idèle class character. Fix  $s_0 \in \mathbb{C}$  with  $\Re(s_0) \geq 1$  and assume that the Eisenstein series  $E(g, f_{\tau_s})$  introduced in (4.6) has a pole at  $s = s_0$ . Then,  $s_0 = 1$  and  $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$  has a simple pole at  $s = 1$  in the odd case while  $L^S(s, \tau, \mathrm{Sym}^2 \otimes \omega^{-1})$  has a simple pole at  $s = 1$  in the even case.*

*Proof.* We know from the general theory of Euler products of Langlands and the Langlands-Shahidi method that the poles of  $E(g, f_{\tau'})$  come from its constant term along  $\mathbf{P}_m$ .

For a decomposable section  $f_{\tau'_s}$  the constant term of  $E(g, f_{\tau'})$  along  $\mathbf{P}_m$  has the form

$$f_{\tau'_s}(I) + \prod_{v \in T} M(f_{\tau'_s}^{(v)}) \frac{L^T(2s-1, \tau, \wedge^2 \otimes \omega^{-1})}{L^T(2s, \tau, \wedge^2 \otimes \omega^{-1})} \quad (5.5)$$

in the odd case, and

$$f_{\tau'_s}(I) + \prod_{v \in T} M(f_{\tau'_s}^{(v)}) \frac{L^T(2s-1, \tau, \text{Sym}^2 \otimes \omega^{-1})}{L^T(2s, \tau, \text{Sym}^2 \otimes \omega^{-1})} \quad (5.6)$$

in the even case, where  $T$  is a finite set of places of  $k$  containing  $S$ .

We should recall that in the constructing the Eisenstein series we used  $s - \frac{1}{2}$  in (4.4) instead of the usual  $s$ . This is responsible for the appearance of  $2s-1$  and  $2s$  instead of the usual  $2s$  and  $2s+1$  in the constant term. Furthermore, the terms  $2s$  and  $2s+1$  appear because we have

$$\text{Ind}_{\mathbf{P}_m(\mathbb{A})}^{\mathbf{H}(\mathbb{A})}(\tau | \det|^s \otimes \omega) = \text{Ind}_{\mathbf{P}_m(\mathbb{A})}^{\mathbf{H}(\mathbb{A})}(2s\tilde{\alpha}, \tau \otimes \omega), \quad (5.7)$$

where the right hand side is as in (4.5) and  $\tilde{\alpha}$  on the left hand side is the notation from the Langlands-Shahidi method.

The terms  $M(f_{\tau'_s}^{(v)})$ , the local intertwining operators at  $I$ , are holomorphic for  $\Re(s) \geq 1$  for all  $v$  [Sh2, Sh3]. Therefore, if  $E(g, f_{\tau,s})$  has a pole at  $s = s_0$ , then

$$\frac{L^T(2s-1, \tau, \wedge^2 \otimes \omega^{-1})}{L^T(2s, \tau, \wedge^2 \otimes \omega^{-1})} \quad (5.8)$$

has a pole at  $s = s_0$  in the odd case, or

$$\frac{L^T(2s-1, \tau, \text{Sym}^2 \otimes \omega^{-1})}{L^T(2s, \tau, \text{Sym}^2 \otimes \omega^{-1})} \quad (5.9)$$

has a pole at  $s = s_0$  in the even case.

Now assume that  $E(\cdot, f_{\tau,s})$  does have a pole at  $s = s_0$  with  $\Re(s_0) \geq 1$ . Then  $\Re(2s_0) \geq 2$  and by [KS2, Prop. 7.3] the denominator in both (5.8) and (5.9) is non-vanishing for  $\Re(s) \geq 1$ . Therefore, the numerator has a pole at  $s = s_0$ . Because  $\Re(2s_0 - 1) \geq 1$  Lemma 5.3 implies that  $s_0 = 1$  and the proof is complete.  $\square$

**Theorem 5.10.** *Let  $\pi$  be an irreducible, unitary, cuspidal, globally generic representation of  $\mathbf{G}(n, \mathbb{A})$ . Let  $\tau$  be an irreducible, unitary, cuspidal representation of  $\text{GL}(m, \mathbb{A})$  with  $2 \leq m \leq n$ . Assume that  $S$  is a sufficiently large finite set of places including all the archimedean places of  $k$ .*

- (a) *The  $L$ -function  $L^S(s, \pi \times \tau)$  is holomorphic for  $\Re(s) > 1$ .*
- (b) *Let  $\omega$  be an idèle class character. Assume that  $\tau \cong \tilde{\tau} \otimes \omega$ . If  $L^S(s, \sigma \times \tau)$  has a pole at  $s = 1$ , then  $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  in the odd case and  $L^S(s, \tau, \text{Sym}^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  in the even case. Such a pole would be simple.*

*When  $m = 1$ , the  $L$ -function  $L^S(s, \sigma \times \tau)$  is entire in both cases.*

*Proof.* Assume that  $L^S(s, \sigma \times \tau)$  has a pole at  $s = s_0$  with  $\Re(s_0) \geq 1$ . By [KS2, Prop. 7.3] we know that both  $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$  and  $L^S(s, \tau, \text{Sym}^2 \otimes \omega^{-1})$  are holomorphic and non-vanishing at  $s = 2s_0$ . Hence, the right hand side of (4.19) has a pole at  $s = s_0$ . Theorem 4.18 then implies that  $I(\phi, f_{\tau'_s})$  has a pole at  $s = s_0$ . Here  $\tau'_s$  is defined in terms of  $\tau$  and  $\omega$  as in (4.4).



Consequently, the Eisenstein series  $E(g, f_{\tau_s})$  must have a pole at  $s = s_0$ . Proposition 5.4 now implies that  $s_0 = 1$  and  $L^S(s, \tau, \wedge^2 \otimes \omega^{-1})$ , in the odd case, or  $L^S(s, \tau, \text{Sym}^2 \otimes \omega^{-1})$ , in the even case, has a simple pole at  $s = 1$ . This proves (a) and (b).

Finally, if  $m = 1$ , then the left hand side of (4.19) is entire, which implies that  $L^S(s, \sigma \times \tau)$  is entire, too. This completes the proof.  $\square$

**Theorem 5.11.** *Let  $\pi$  be an irreducible, automorphic, unitary, cuspidal, globally generic representation of  $\mathbf{G}(n, \mathbb{A})$  with central character  $\omega = \omega_\pi$  and let  $\Pi$  be a transfer of  $\pi$  to  $\text{GL}(2n, \mathbb{A})$ . Assume that  $\Pi$  is a subquotient of  $\Sigma$  as in (5.1) with  $n_1 + n_2 + \cdots + n_t = 2n$ .*

- (a) *We have  $r_1 = r_2 = \cdots = r_t = 0$ .*
- (b) *The representations  $\sigma_i$  are pairwise inequivalent,  $n_i \geq 2$ , and  $\sigma_i \cong \tilde{\sigma}_i \otimes \omega$  for all  $i$ . Moreover, for  $S$  a sufficiently large finite set of places including all the archimedean ones, we have that  $L^S(s, \sigma_i, \wedge^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  in the odd case, and  $L^S(s, \sigma_i, \text{Sym}^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  in the even case.*

*Proof.* By [AS1, Prop. 7.4] we know that  $\Sigma$  is induced from a representation of a Levi subgroup of  $\text{GL}(2n)$  of type  $(a_1, \dots, a_q, b_1, \dots, b_\ell, a_q, \dots, a_1)$  which can be written as

$$\begin{aligned} & \delta_1 |\det(\cdot)|^{z_1} \otimes \cdots \otimes \delta_q |\det(\cdot)|^{z_q} \otimes \\ & \quad \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_\ell \\ & \quad \otimes (\tilde{\delta}_q \otimes \omega^{-1}) |\det(\cdot)|^{-z_q} \otimes \cdots \otimes (\tilde{\delta}_1 \otimes \omega^{-1}) |\det(\cdot)|^{-z_1}, \end{aligned}$$

where  $\delta_i$  and  $\sigma_i$  are irreducible, unitary, cuspidal presentations,  $\sigma_i \cong \tilde{\sigma}_i \otimes \omega$ , and

$$2(a_1 + \cdots + a_q) + (b_1 + \cdots + b_\ell) = 2n. \quad (5.12)$$

Assume that  $q > 0$ . Rearranging if necessary we may assume  $\Re(z_1) \leq \cdots \leq \Re(z_q) < 0$ . Now for  $S$  a sufficiently large finite set of places we have

$$\begin{aligned} L^S(s, \pi \times \tilde{\delta}_1) &= L^S(s, \Pi \times \tilde{\delta}_1) \\ &= \prod_{i=1}^q L^S(s + z_i, \delta_i \times \tilde{\delta}_1) L^S(s - z_i, \tilde{\delta}_i \times \tilde{\delta}_1 \otimes \omega^{-1}) \\ &\quad \cdot \prod_{i=1}^\ell L^S(s, \sigma_i \times \tilde{\delta}_1). \end{aligned} \quad (5.13)$$

The first term on the right hand side has a pole at  $s = 1 - z_1$  which can not be canceled by the other terms because  $\Re(1 - z_1 \pm z_i) \geq 1$  and  $\Re(1 - z_1) \geq 1$ . Therefore, the left hand side has a pole at  $s = 1 - z_1$ .

On the other hand, by (5.12) we know that  $a_1 \leq n$ . We can apply Theorem 5.10(a) to conclude that  $\Re(z_1) \geq 0$ . This is a contradiction proving that  $q = 0$ , i.e., there are no  $\delta_i$ 's.

So far we have proved that  $\Sigma$  is induced from a representation of the form  $\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_\ell$  satisfying  $\sigma_i \cong \tilde{\sigma}_i \otimes \omega$ . Fix  $1 \leq j \leq \ell$  and consider

$$L^S(s, \pi \times \tilde{\sigma}_j) = L^S(s, \Pi \times \tilde{\sigma}_j) = \prod_{i=1}^\ell L^S(s, \sigma_i \times \tilde{\sigma}_j). \quad (5.14)$$

The right hand side has a pole and hence, so does the left hand side. Moreover, the  $\sigma_i$ 's are pairwise inequivalent because otherwise the left hand side of (5.14) would have a pole of higher order. Since  $\sigma_i \cong \tilde{\sigma}_i \otimes \omega$  we can apply Theorem 5.10(b). We conclude that  $L^S(s, \sigma_i, \wedge^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  in the odd case, and  $L^S(s, \sigma_i, \text{Sym}^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  in the even case.

Finally, Theorem 5.10(c) shows that no  $\sigma_j$  is a character. This completes the proof.  $\square$

**Corollary 5.15.** *The representation  $\Sigma$  is irreducible and  $\Pi = \Sigma = \sigma_1 \boxplus \cdots \boxplus \sigma_t$  is an isobaric sum of the  $\sigma_i$ . In particular, the transfer  $\Pi$  of  $\pi$  is unique and  $\Pi \cong \tilde{\Pi} \otimes \omega$  (not just nearly equivalent as in [AS1, Theorem 1.1]).*

*Proof.* The corollary immediately follows from the fact that  $r_1 = \cdots = r_t = 0$  and that  $\sigma_i \cong \tilde{\sigma}_i \otimes \omega$ .  $\square$

**5.2. Description of the Image of Transfer.** We continue to denote by  $\pi$  an irreducible, globally generic, unitary, cuspidal automorphic representation of  $\mathbf{G}(n, \mathbb{A})$ . We proved that  $\pi$  has a unique transfer  $\Pi$ , an irreducible, generic, automorphic representation of  $\mathrm{GL}(2n, \mathbb{A})$ . Moreover, we have shown that  $\omega_\Pi = \omega^n \mu$  and  $\Pi \cong \tilde{\Pi} \otimes \omega$ , where  $\omega = \omega_\pi$  denotes the central character of  $\pi$  and  $\omega_\Pi$  denotes that of  $\Pi$ .

Furthermore, Theorem 5.11 gives an “upper bound” for the image of transfer from  $\mathbf{G}(n)$  groups to  $\mathrm{GL}(2n)$ . Combining this with the “lower bound” provided by Hundley and Sayag in [HS1, HS2, HS3] gives the full description of the image of this transfer. We summarize all these results as follows.

**Theorem 5.16.** *Let  $k$  be a number field and let  $\mathbb{A} = \mathbb{A}_k$  be the ring of adèles of  $k$ . Denote by  $\mathbf{G}(n)$  the split groups  $\mathrm{GSpin}(2n+1)$ ,  $\mathrm{GSpin}(2n)$ , or any of the quasi-split non-split groups  $\mathrm{GSpin}^*(2n)$ . Let  $\pi$  be a globally generic, irreducible, cuspidal, automorphic representation of  $\mathbf{G}(n, \mathbb{A})$  with central character  $\omega = \omega_\pi$ . Then  $\pi$  has a unique functorial transfer to an automorphic representation  $\Pi$  of  $\mathrm{GL}(2n, \mathbb{A})$  associated with the  $L$ -homomorphism  $\iota$  described [AS1] (split case) and Section 3 (quasi-split non-split). The transfer  $\Pi$  satisfies*

$$\Pi \cong \tilde{\Pi} \otimes \omega.$$

Moreover,

$$\omega_\Pi = \omega_\pi^n \mu,$$

where  $\mu$  is a quadratic idèle class character which is trivial in the split case and nontrivial in the quasi-split non-split case. (The triviality or nontriviality of  $\mu = \omega_\Pi \omega^{-n}$  can tell apart the split and quasi-split non-split cases.)

The automorphic representation  $\Pi$  is an isobaric sum of the form

$$\Pi = \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_t) = \Pi_1 \boxplus \cdots \boxplus \Pi_t,$$

where each  $\Pi_i$  is a unitary, cuspidal representation of  $\mathrm{GL}(n_i, \mathbb{A})$  such that for  $T$  sufficiently large finite set of places of  $k$  containing the archimedean places, the partial  $L$ -function  $L^T(s, \Pi_i, \wedge^2 \otimes \omega)$  has a pole at  $s = 1$  in the odd case and  $L^T(s, \Pi_i, \mathrm{Sym}^2 \otimes \omega)$  has a pole at  $s = 1$  in the even case (both split and quasi-split non-split cases). We have  $\Pi_i \not\cong \Pi_j$  if  $i \neq j$  and  $n_1 + \cdots + n_t = 2n$  with each  $n_i > 1$ .

Furthermore, any such representation  $\Pi$  is a functorial transfer of some globally generic  $\pi$ .

## 6. APPLICATIONS

**6.1. Local Representations at the Ramified Places.** The local components of the automorphic representation  $\Pi = \otimes_v \Pi_v$  are well understood for the archimedean  $v$  as well as those non-archimedean  $v$  outside of the finite set  $S$  through our construction of the candidate transfer. However, the converse theorem tells us nothing about  $\Pi_v$  for  $v \in S$ . Having proved Theorem 5.16 we can now get some information for these places as well. This shows that while we did not have control over places  $v \in S$ , the automorphic representation  $\Pi$  does indeed turn out to have the right local components in  $S$ .

**Proposition 6.1.** *Let  $S$  be the non-empty finite set of non-archimedean places as in [AS1, Thm. 1.1.] or Theorem 3.3. Fix  $v \in S$  and let*

$$\pi_v \cong \text{Ind} \left( \pi_{1,v} | \det |^{b_{1,v}} \otimes \cdots \otimes \pi_{r,v} | \det |^{b_{r,v}} \otimes \pi_{0,v} \right)$$

*be an irreducible, generic representation of  $\mathbf{G}(n, k_v)$ , where each  $\pi_{i,v}$  is a tempered representation of  $\text{GL}(n_i, k_v)$ ,  $b_{1,v} > \cdots > b_{r,v}$  and  $\pi_{0,v}$  is a tempered, generic representation of some smaller  $\mathbf{G}(m, k_v)$  with  $n_1 + \cdots + n_r + m = n$ . Denote the central character of  $\pi_v$  by  $\omega_v$ .*

*Assume that  $\pi_v$  is the local component of the globally generic representation  $\pi$  of  $\mathbf{G}(n, \mathbb{A}_k)$  and let  $\Pi$  be its transfer to  $\text{GL}(2n, \mathbb{A}_k)$ . Then the local component  $\Pi_v$  of  $\Pi$  at  $v$  is generic and of the form*

$$\Pi_v \cong \text{Ind} \left( (\pi_{1,v} | \det |^{b_{1,v}} \otimes \cdots \otimes \pi_{r,v} | \det |^{b_{r,v}} \otimes \Pi_{0,v} \otimes \right. \quad (6.2)$$

$$\left. (\tilde{\pi}_{r,v} \otimes \omega_v) | \det |^{-b_{r,v}} \otimes \cdots \otimes (\tilde{\pi}_{1,v} \otimes \omega_v) | \det |^{-b_{1,v}} \right), \quad (6.3)$$

*where  $\Pi_{0,v}$  is a tempered representation of  $\text{GL}(2m, k_v)$  if  $m > 0$ .*

*Proof.* The argument proceeds the same way as in the proof of [AS2, Prop. 2.5.], which proved the analogous result for the case of  $\text{GSp}(4) = \text{GSpin}(5)$ . We briefly mention the steps for completeness.

The fact that  $\Pi$  is an isobaric sum of unitary, cuspidal representations of general linear groups, Theorem 5.16, implies that every local component of  $\Pi$  is full induced and generic. In particular, so is  $\Pi_v$ .

The first step is to show that

$$\gamma(s, \pi_v \times \rho_v, \psi_v) = \gamma(s, \Pi_v \times \rho_v, \psi_v)$$

for every supercuspidal representation  $\rho_v$  of  $\text{GL}(a, k_v)$ . To do this we “embed” the local representation  $\rho_v$  in a unitary, cuspidal representation  $\rho$  of  $\text{GL}(a, \mathbb{A})$  whose other local components are unramified [Sh3, Prop. 5.1] and apply the converse theorem with  $S' = S - \{v\}$  just as in [CKPSS2, Prop. 7.2]. Moreover, by multiplicativity of the  $\gamma$ -factors we obtain the equality for  $\rho_v$  in the discrete series, as well.

Next, assume that  $\pi_v$  is tempered. We claim that  $\Pi_v$  is also tempered. Here again the main tool is multiplicativity of the  $\gamma$ -factors and the proof is exactly as in [CKPSS2, Lemma 7.1]. This proves the Proposition for  $r = 0$ .

Now consider the case of  $r > 0$ . Apply the case of  $r = 0$  to  $\pi_v = \pi_{0,v}$  and take the resulting tempered representation of  $\text{GL}(2m, k_v)$  to be  $\Pi_{0,v}$ . To show that this representation satisfies the requirements of the proposition we use the converse theorem again. Let  $T = \{w\}$  consist of a single non-archimedean place  $w \neq v$  at which  $\pi_v$  is unramified and consider the global representation  $\Pi'$  of  $\text{GL}(2n, \mathbb{A})$  whose local components are the same as those of  $\Pi$  outside of  $S$  and are the irreducible, induced representations on the right hand side of (6.2) when  $v \in S$ . We can apply the converse theorem, Theorem 3.5, to  $\Pi'$  and  $T$  because the induced representations on the right hand side of (6.2) have the right local  $L$ -functions. The conclusion is that  $\Pi'$  is a transfer of  $\pi$  (outside of  $T$ ) and by the uniqueness of the transfer, Theorem 5.16, we have  $\Pi'_v \cong \Pi_v$  for  $v \in S$ . This completes the proof.  $\square$

**6.2. Ramanujan Estimates.** Following [CKPSS2], we introduce the following notation. Let  $\Pi = \otimes_v \Pi_v$  be a unitary, cuspidal, automorphic representation of  $\text{GL}(m, \mathbb{A}_k)$ . For each place  $v$  the representation  $\Pi_v$  is unitary generic and can be written as a full induced representation

$$\Pi_v \cong \text{Ind} \left( \Pi_{1,v} | \det |^{a_{1,v}} \otimes \cdots \otimes \Pi_{r,v} | \det |^{a_{r,v}} \right)$$

with  $a_{1,v} > \cdots > a_{r,v}$  and each  $\Pi_{i,v}$  tempered.

**Definition 6.4.** We say  $\Pi$  satisfies  $H(\theta_m)$  with  $\theta_m \geq 0$  if for all places  $v$  we have  $-\theta_m \leq a_{i,v} \leq \theta_m$ .

The classification of the generic unitary dual of  $\mathrm{GL}(m)$ , [Td, V], trivially gives  $H(1/2)$ . The best result currently known for a general number field is  $\theta_m = 1/2 - 1/(m^2 + 1)$  proved in [LRS] with a few better results known for small values of  $m$  and over  $\mathbb{Q}$ . The Ramanujan conjecture for  $\mathrm{GL}(m)$  demands  $H(0)$ .

Similarly, if  $\pi = \otimes_v \pi_v$  is a unitary, generic, cuspidal, automorphic representation of  $\mathbf{G}(n, \mathbb{A}_k)$  each  $\pi_v$  can be written as a full induced representation

$$\pi_v \cong \mathrm{Ind} \left( \pi_{1,v} | \det |^{b_{1,v}} \otimes \cdots \otimes \pi_{r,v} | \det |^{b_{r,v}} \otimes \tau_v \right),$$

where each  $\pi_{i,v}$  is a tempered representation of some  $\mathrm{GL}(n_i, k_v)$  and  $\tau_v$  is a tempered, generic representation of some  $\mathbf{G}(m, k_v)$  with  $n_1 + \cdots + n_t + m = n$ .

**Definition 6.5.** We say  $\pi$  satisfies  $H(\theta_n)$  with  $\theta_n \geq 0$  if for all places  $v$  we have  $-\theta_m \leq b_{i,v} \leq \theta_m$ .

Again, we would have the bound  $H(1)$  trivially as a consequence of the classification of the generic unitary dual and the Ramanujan conjecture demands  $H(0)$ .

**Proposition 6.6.** Let  $k$  be a number field and assume that all the unitary, cuspidal representations of  $\mathrm{GL}(m, \mathbb{A}_k)$  satisfy  $H(\theta_m)$  for  $2 \leq m \leq 2n$  and  $\theta_2 \leq \theta_3 \leq \cdots \leq \theta_{2n}$ . Then any globally generic, unitary, cuspidal representation  $\pi$  of  $\mathbf{G}(n, \mathbb{A}_k)$  satisfies  $H(\theta_{2n})$ . In fact, if  $\pi$  transfers to a non-cuspidal representation  $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t$ , then  $\pi$  satisfies the possibly better bound of  $H(\theta)$  where  $\theta = \max\{\theta_{n_1}, \theta_{n_2}, \dots, \theta_{n_t}\}$ . Here,  $\Pi_i$  is a unitary, cuspidal representation of  $\mathrm{GL}(n_i, \mathbb{A}_k)$ .

*Proof.* The argument is exactly the same as the proof of [AS2, Theorem 3.3] and we do not repeat it here. Note that our Proposition 6.1 is used for the ramified non-archimedean places.  $\square$

**Corollary 6.7.** Every globally generic, unitary, cuspidal, automorphic representation  $\pi$  of  $\mathbf{G}(n, \mathbb{A}_k)$  satisfy

$$H \left( \frac{4n^2 - 1}{2(4n^2 + 1)} \right).$$

If  $\pi$  transfers to a non-cuspidal, automorphic representation  $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_t$ , then we can replace  $n$  with the size of the largest  $\mathrm{GL}$  block appearing, resulting in a better estimate.

*Proof.* This is immediate if we combine Proposition 6.6 with the  $\mathrm{GL}(m)$  estimate of  $1/2 - 1/(m^2 + 1)$ .  $\square$

We should remark that for small values of  $n$  it is possible to obtain better estimates because much better estimates are available for small general linear groups (and also for  $k = \mathbb{Q}$ ). For an example, see [AS2, §3.1]

**Corollary 6.8.** The Ramanujan conjecture for the unitary, cuspidal representations of  $\mathrm{GL}(m, \mathbb{A}_k)$  for  $m \leq 2n$  implies the Ramanujan conjecture for the generic spectrum of  $\mathbf{G}(n, \mathbb{A}_k)$ .

*Proof.* This is an immediate corollary of Proposition 6.6 where all the  $\theta$ 's are zero.  $\square$

**6.3. Image of Kim’s exterior square.** H. Kim proved the exterior square transfer of automorphic representations from  $\mathrm{GL}(4, \mathbb{A}_k)$  to  $\mathrm{GL}(6, \mathbb{A}_k)$  [K2, H2]. A. Raghuram and the first author gave a complete cuspidality criterion for this transfer, determining when the image of this transfer is not cuspidal [AR]. A natural question about the image of this transfer is which automorphic representations of  $\mathrm{GL}(6, \mathbb{A}_k)$  are indeed in the image of this transfer. We can now answer this question as an application of our Theorem 5.16.

**Proposition 6.9.** *Let  $\Pi$  be a cuspidal, automorphic representation of  $\mathrm{GL}(6, \mathbb{A}_k)$ . Then there is a globally generic, cuspidal, automorphic representation  $\pi$  of  $\mathrm{GL}(4, \mathbb{A}_k)$  such that  $\Pi = \wedge^2 \pi$  if and only if there is an idèle class character  $\omega$  such that the partial  $L$ -function  $L^S(s, \Pi, \mathrm{Sym}^2 \otimes \omega^{-1})$  has a pole at  $s = 1$  for  $S$  a sufficiently large finite set of places of  $k$  including all the archimedean ones.*

*Proof.* The proposition follows immediately from our Theorem 5.16 if we recall that Kim’s exterior square transfer from  $\mathrm{GL}(4)$  to  $\mathrm{GL}(6)$  is a special case of the transfer in the split even case of our theorem when  $m = 3$ , i.e., the transfer from  $\mathrm{GSpin}(6)$  to  $\mathrm{GL}(6)$  [AS1, Prop. 7.6].

If we assume that  $\Pi$  is the transfer of  $\pi$ , then we have proved that we can take  $\omega = \omega_\pi$ , the central character of  $\pi$ . The opposite direction requires the descent method in our cases and would follow from J. Hundley and E. Sayag’s “lower bound” result for our transfer [HS1, HS2, HS3] because Kim’s  $\wedge^2$  is a special case of transfer from  $\mathrm{GSpin}(6)$  to  $\mathrm{GL}(6)$  as mentioned above.  $\square$

Another natural question regarding the image of Kim’s exterior square transfer is to determine “the fiber” for each cuspidal  $\Pi$  which is indeed in the image. In other words, determine all representations  $\pi$  such that  $\Pi = \wedge^2 \pi$ .

A further interesting question would be to explore possible overlaps between various transfers to cuspidal representations of  $\mathrm{GL}(6)$ . As pointed out in [CPSS2, §6] (for the untwisted  $\omega = 1$  case) and as it is apparent from our Theorem 5.16 there can be no overlap between the images of transfers from  $\mathrm{GSpin}(7)$  and quasi-split forms of  $\mathrm{GSpin}(6)$  (which includes Kim’s transfer) to  $\mathrm{GL}(6)$ . However, there may be potential overlaps with the transfer from unitary groups or the Kim-Shahidi transfer [KS1] from  $\mathrm{GL}(2) \times \mathrm{GL}(3)$  to  $\mathrm{GL}(6)$ .

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